THE INFLUENCE OF ANOMALOUS GRAVITY ON THE PERFORMANCE OF A MECHANICALLY PERFECT INERTIAL NAVIGATION SYSTEM

by

Mark Morris Macomber, B.S., M.Sc.

The Ohio State University
1966

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Approved by

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CONTENTS

FIGURES

CHAPTER I

BASIC CONCEPTS

The concept of inertial navigation is not new. Sir Isaac Newton, in his "Mathematical Principles of Natural Philosophy," gave the laws of mechanics on which self-contained inertial navigation is based. At the time of the publication of Newton's treatise there were no means available for sensing accelerations accurately or for economically determining positions if sensing had been possible. It has taken almost three hundred years to develop the necessary hardware and software to make self-contained inertial navigation a reality.

A brief description of an inertial navigation system is that we have a "black box" which senses both accelerations and mass attraction. We eliminate the effect of the mass attraction and then use our initial position and velocity, elapsed time, and the double integral of the acceleration to determine an updated position. In practice this is done many times a second, providing a continuous readout of incrementally corrected position. Henry described part of the mechanism in the following words:

> A basic requirement of an inertial navigation system is that the acceleration vector be known at all times. Knowledge of the direction of this vector requires either maintaining an acceleration sensitive element in a fixed orientation or continuous knowledge of the orientation assumed by the acceleration sensitive element.

> The system may take any one of the various forms in which the acceleration sensitive element or elements are maintained relative to the local gravity vector (geodetic level) or geocentric level (that is, pointed towards the center of mass of the earth). The stable element may be fixed with respect to the earth, rotating around the earth's axis at earth rate, or may simply be fixed in inertial space. An azimuth line on a level platform may be oriented to point at geodetic north or in some other arbitrary direction. (4)

During the 1950's and 1960's a tremendous effort has been devoted to eliminating errors in sensing systems and servo systems so that the acceleration sensitive elements may be maintained in a known orientation. As stated by Moore:

> The development of inertial navigators has been one of the major technological achievements of the twentieth century. The outward simplicity of the mechanization of these navigation systems and the ease of statement of the simple, fundamental scientific principles on which they are based mask a major scientific engineering and manufacturing development program. It has taken the combined efforts of engineering specialists in electronics, mechanical, chemical, and electrical engineering as well as physicists, mathematicians, metallurgists, metrologists, skilled machinists, and experienced electronics and electromechanical manufacturing departments working in close coordination to develop and produce the apparently simple end product. (6)

It has become apparent, in searching the literature, that some degree of confusion exists about our ability to eliminate the effect of mass attraction from the values sensed by the "black box." Statements such as that of Henry, quoted above, equating the gravity vector with the normal to geodetic level are common. In those cases where some acknowledgment is made that the magnitude and direction of gravity do not agree with simple formulas, the effect is considered as random noise in the system.

It is the purpose of this paper to assess the impact of the

variations of gravity on a vessel, such as a survey ship, which is likely to remain within a limited area for an extended period of time, such that the effects of anomalous gravity are systematic rather than random, and to suggest some system to eliminate these effects from the navigation solution.

The effects of gravity on the three forms of the system as described above by Henry will be investigated to see if the different forms are affected differently. These three forms, in order of increasing complexity, are:

a. A system in which the orientation of the sensor axes is fixed in inertial space,

b. A system in which the orientation of the sensor axes maintains a fixed relationship to the earth, with the acceleration sensitive elements rotating at earth rate about an axis which is parallel to the earth's spin axis, and

c. A system in which the orientation of the sensor axes is dependent on the navigator's position, where one axis is aligned with the normal to the adopted ellipsoid at that position, one axis points north in the plane tangent to the ellipsoid, and the third axis completes an orthogonal system.

Three coordinate systems will be employed in this study, all of them being right-handed orthogonal systems. The X system is an earth-fixed system, with the X^3 axis parallel to the earth's rotational axis, positive to the north, the X^0 axis parallel to the

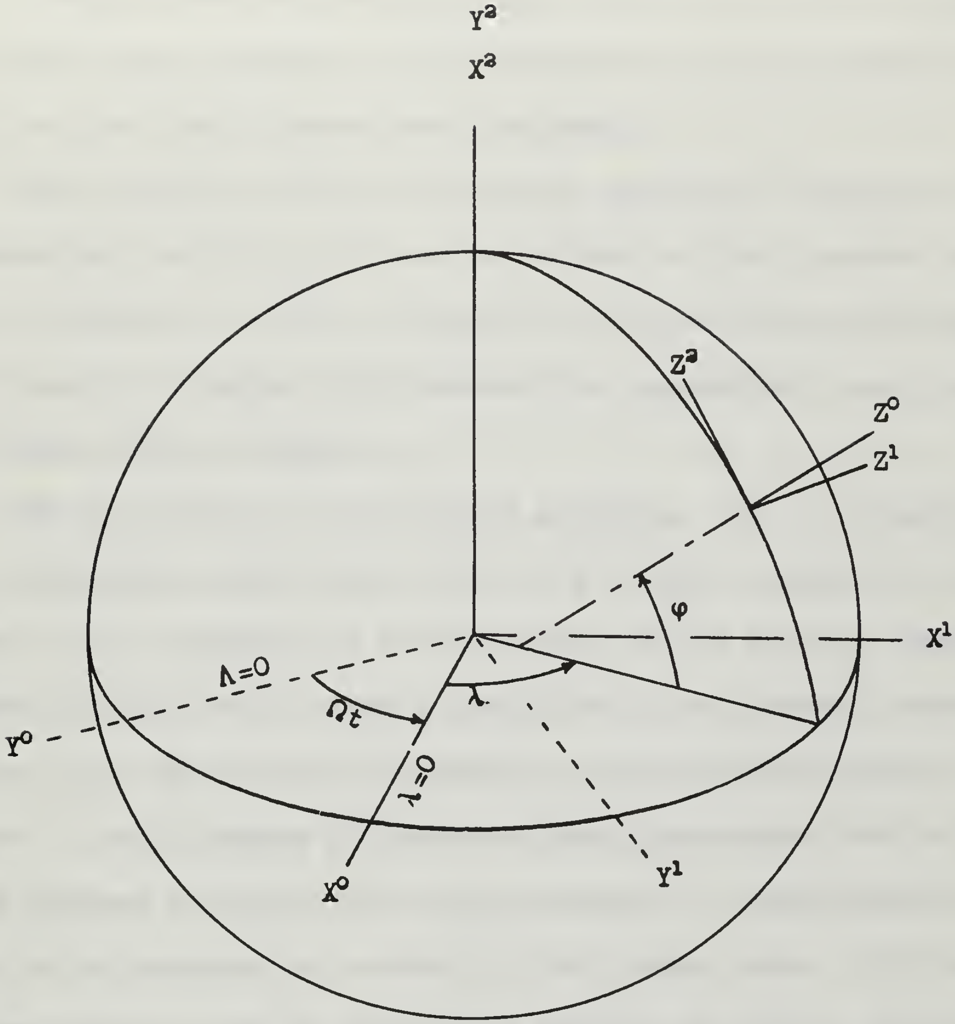

Figure 1.

Relationships between coordinate systems

equatorial plane and the Greenwich meridional plane, positive in the direction from the center of the earth to the zero meridian, and the X^1 axis positive in the direction from the center of the earth to the ninety degree east meridian.

The Y system is fixed in inertial space with respect to orientation, with $Y^2 = X^2$, and is related to the X system such that, at time $t = 0$, $Y^0 = X^0$ and $Y^1 = X^1$, and at any other time there exists an angle of Ωt between the respective Y and X axes in the sense shown in figure 1.

The Z system is a function of position, and is oriented to the ellipsoidal normal such that the Z^0 axis is parallel to the normal and is positive in the direction of the outward normal (generally up), the Z^1 axis is positive in the geodetic east direction, and the Z^2 axis is positive toward geodetic north. In figure 1, the Z system is shown as being topocentric while the X and Y systems are shown as being geocentric. This distinction was made in the interest of clarity of the figure only. All three systems will be used as geocentric systems so that no translation between origins is required.

Rotation matrices used to rotate the coordinate systems about any coordinate axis will be of the form $R_j(q)$ where the index "j" indicates the axis about which the rotation "q" takes place, with the sense of "q" in accordance with the right-hand rule. Thus $\vec{Y} = R_2(-\Omega t)\vec{X}$ and $\vec{X} = R_2(-\lambda)R_1(\varphi)\vec{Z}$ where the position vectors are defined as columns.

The following symbols will be used throughout this study:

a - semimajor axis of the ellipsoid

b - semiminor axis of the ellipsoid

e - eccentricity of the meridian ellipse = $(a^2 - b^2)^{1/2}/a$

$\vec{g}$ - gravity vector

h - ellipsoidal height

$\vec{i}$ - unit vector along the zero axis of a coordinate system

$\vec{j}$ - unit vector along the one axis of a coordinate system

$\vec{k}$ - unit vector along the two axis of a coordinate system

k - Newtonian constant of gravitation

$\vec{G}$ - gravitation vector

H - orthometric height

M - radius of curvature in the meridian = $a(1-e^2)/W^3$

N - radius of curvature in the prime vertical = a/W

- as a subscript, $_N$, indicates navigator

$\vec{R}$ - radius vector

W - $(1 - e^2 \sin^2 \varphi)^{1/2}$

α - azimuth

γ - value of normal gravity

ζ - geoid undulation

η - prime vertical component of deflection of the vertical

λ - geodetic longitude, positive east, measured from X^0 axis

λ' - astronomic longitude, positive east

ξ - meridional component of deflection of the vertical

φ - geodetic latitude, positive north

$\bar{\varphi}$ - geocentric latitude, positive north

φ' - astronomic latitude, positive north

$\vec{\omega}$ - generalized rotation vector

Λ - inertial longitude, positive east, measured from the Y^o axis $= \lambda + \Omega t$

$\vec{\Omega}$ - earth angular velocity vector

$\dot{()}$ - first derivative of () with respect to time

$\ddot{()}$ - second derivative of () with respect to time

δg - gravity disturbance

Δg - gravity anomaly

$\vec{()}^o$ - unit vector in the direction of $\vec{()}$

Vector quantities, other than i, j, k, when written without the sign of the vector, $\rightarrow$, denote magnitude of the quantity considered.

There are several quantities used in this study which require specific definition in order to avoid a chance of confusion.

Meridian plane - The plane containing the normal to the ellipsoid at a given point and the axis of rotation of the ellipsoid. The meridian is the intersection of the meridian plane and the ellipsoidal surface for that half of the ellipse which contains the given point, and is bounded by the intersection of the ellipse with the rotational axis. Meridian and meridian plane are frequently used interchangeably.

Prime vertical plane - The plane containing the normal to the ellipsoid at a given point which is perpendicular to the meridian plane at that point. The prime vertical is the intersection of the prime vertical plane and the ellipsoidal surface. It is tangent to the parallel of latitude at the given point. Prime

vertical and prime vertical plane are often used interchangeably.

Latitude, geodetic - φ - the angle between the normal to the ellipsoid at a given point and a plane perpendicular to the axis of rotation of the ellipsoid.

Latitude, astronomic - φ' - The angle between the gravity vector at a given point and a plane perpendicular to the mean rotational axis of the earth.

Latitude, navigator's - φ_N - The value which the inertial navigator indicates as being descriptive of one component of its position.

Latitude, geocentric - $\bar{\varphi}$ - The angle between the radius vector to a given point and a plane perpendicular to the axis of rotation of the ellipsoid.

North error - The difference between either the geodetic or astronomic latitude and the navigator's latitude, converted to a linear measurement. Whether the error is computed for the geodetic or astronomic latitude will be indicated in the text.

North error = $(\varphi_N - \varphi)M$, or North error = $(\varphi_N - \varphi')M$

Longitude, geodetic - λ - The angle between the meridian plane at a given point and the meridian plane of Greenwich.

Longitude, astronomic - λ' - The angle between the astronomic meridian plane of a given point, and the astronomic meridian plane of Greenwich, where the astronomic meridian plane is defined as that plane containing the gravity vector at a given point which is parallel to the axis of rotation of the earth.

Longitude, navigator's - λ_N - The value which the inertial

navigator indicates as being descriptive of a second component of its position, in a direction orthogonal to the navigator's latitude.

<u>East error</u> - The difference between either the geodetic or astronomic longitude and the navigator's longitude, converted to a linear measurement. Whether the error is computed for the geodetic or astronomic longitude will be indicated in the text.
East error = $(\lambda_N - \lambda)N\cos\varphi$, or East error = $(\lambda_N - \lambda')N\cos\varphi$

<u>Deflection of the vertical</u> - ξ, η - The angular distance between the astronomic position, φ', λ', of a point, and the geodetic position, φ, λ, of the same physical point. The deflection is divided into a component in the meridian plane, ξ, and a component in the prime vertical plane, η.

$$\xi = \varphi' - \varphi$$

$$\eta = (\lambda' - \lambda)\cos\varphi$$

<u>Local vertical rate</u> - The time rate of change of the local vertical. This is divided into components in the meridian plane and in the prime vertical plane. For an ellipsoid, the meridional local vertical rate is the rate of change of the ellipsoidal normal in the meridian, or the rate of change of latitude, $\mathring{\varphi}$. For the geoid, we must consider the latitude rate, plus the rate of change of the meridional component of the deflection of the vertical, or $\dot{\varphi} + \mathring{\xi} = \dot{\varphi}'$.

Similarly, the prime vertical local vertical rate for the ellipsoid is $\mathring{\lambda}\cos\varphi$, while for the geoid, the rate of change of the

deflection of the vertical component must be considered, and the prime vertical local vertical rate is $\dot{\lambda}\cos\varphi + \dot{\eta} = \dot{\lambda}'\cos\varphi$. When this term is used in the text it will always refer to the geoidal application rather than the ellipsoidal. The meridional local vertical rate is also referred to as the astronomic latitude rate, while the prime vertical local vertical rate divided by the cosine of the latitude is the astronomic longitude rate.

Gravitation vector - $\vec{G}$ - The force exerted on a body of unit mass because of mass attraction. For a central force field, $\vec{G} = -\frac{km\vec{R}}{R^3}$, where m is the mass of the attracting body.

Gravity vector - $\vec{g}$ - The combined force of gravitation and the centrifugal force caused by the rotation of the earth, when applied to a body of unit mass.

$\vec{g} = \vec{G} - \vec{\Omega}x(\vec{\Omega}x\vec{R})$

Height, ellipsoidal - h - The linear distance, measured along a normal to the ellipsoid, from the ellipsoid to a given point. Height is positive along the outward normal.

Height, orthometric - H - The linear distance, measured along the local vertical, from the geoid to a given point. Height is positive along the outward normals to the geopotential surfaces.

Height, navigator's - h_N - The value which the inertial navigator indicates as being descriptive of the third component of its position, in a direction orthogonal to both the navigator's latitude and the navigator's longitude.

Height error - The difference between the height and the

navigator's height. The two heights will in general be figured at two different horizontal locations.

Height error = $h_N - h$

CHAPTER II

STATIONARY OPERATION OF PERFECT INERTIAL NAVIGATION DEVICES IN A NON-NORMAL FIELD

Conceptually, the simplest form of inertial navigation device is one in which the reference frame maintains a fixed orientation in inertial space, and which contains accelerometers mounted along three mutually perpendicular axes. Given an initial position in space, and an initial velocity, the accelerometer outputs, when properly compensated to remove the effects of mass attraction, are doubly integrated to provide, at any time, the current position. Although the orientation of these mutually perpendicular axes is arbitrary, for ease of consideration we will assume that they correspond to the axes of the Y coordinate system as described in Chapter I.

Equations are developed in Appendix I to permit the determination of acceleration in inertial space from the values sensed by the errorless accelerometers adopted for this study, under the assumption that gravitational forces, which are sensed with accelerations, are normal. This is the assumption used in currently operational navigation devices. A similar development, using hypothetical accelerations and gravitational field, provides us hypothetical values which we consider to be sensed by our errorless accelerometers. After completing the general development,

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we consider the specific case of a navigator who is stationary relative to the earth at a point where the geoid and an adopted reference ellipsoid coincide. The International Ellipsoid was chosen for convenience, even though other ellipsoids are presumably closer to the true size and shape of the geoid. Assumed values of gravity anomaly and deflection of the vertical components were assigned to the hypothetical point to which the navigator was rigidly attached, and these, when inserted into the last equation of Appendix I, provided the apparent motion of the navigator under the influence of non-normal gravitation.

Figures 2, 3, and 4 show the north geodetic error, the east geodetic error and the height error caused by components of non-normal gravitation. The term geodetic used in connection with the word error indicates that the navigator's position is equal to the geodetic position plus the geodetic error. Computations were terminated when the height error reached $\pm$ 20 kilometers, which is well beyond the limits of certain assumptions made in the mathematical approximations employed. It should be noted that the ordinate scale for height error (figure 4) is one-tenth that for the north error and east error, and that once the height error started increasing, it was not a slowly changing function as had been assumed in obtaining the average values of the gravitation function. The results do indicate, however, that this approach is not satisfactory.

If we consider the case where we supplement the inertial navigation information with an independent determination of the

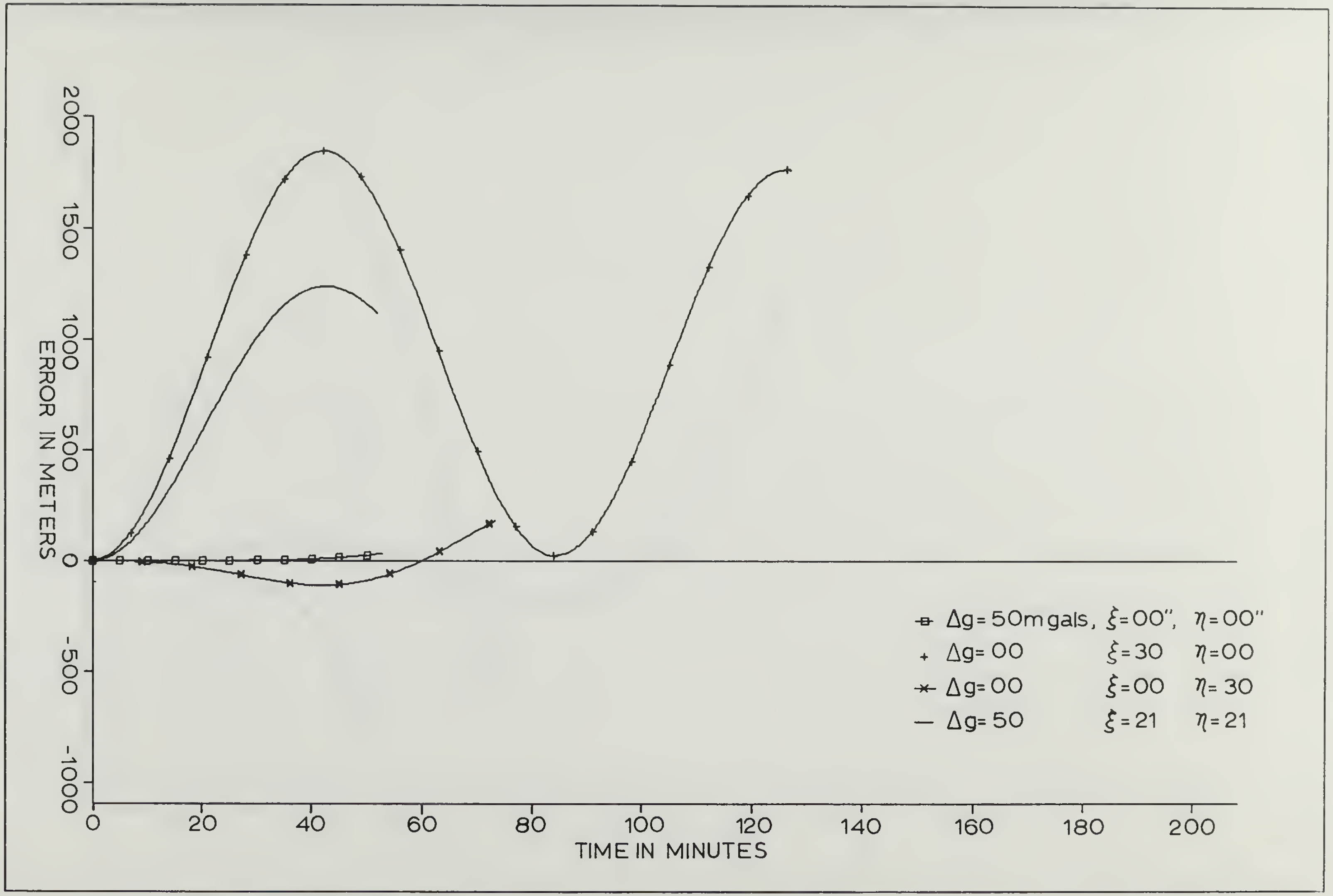


Figure 2. North geodetic error of a stationary observer

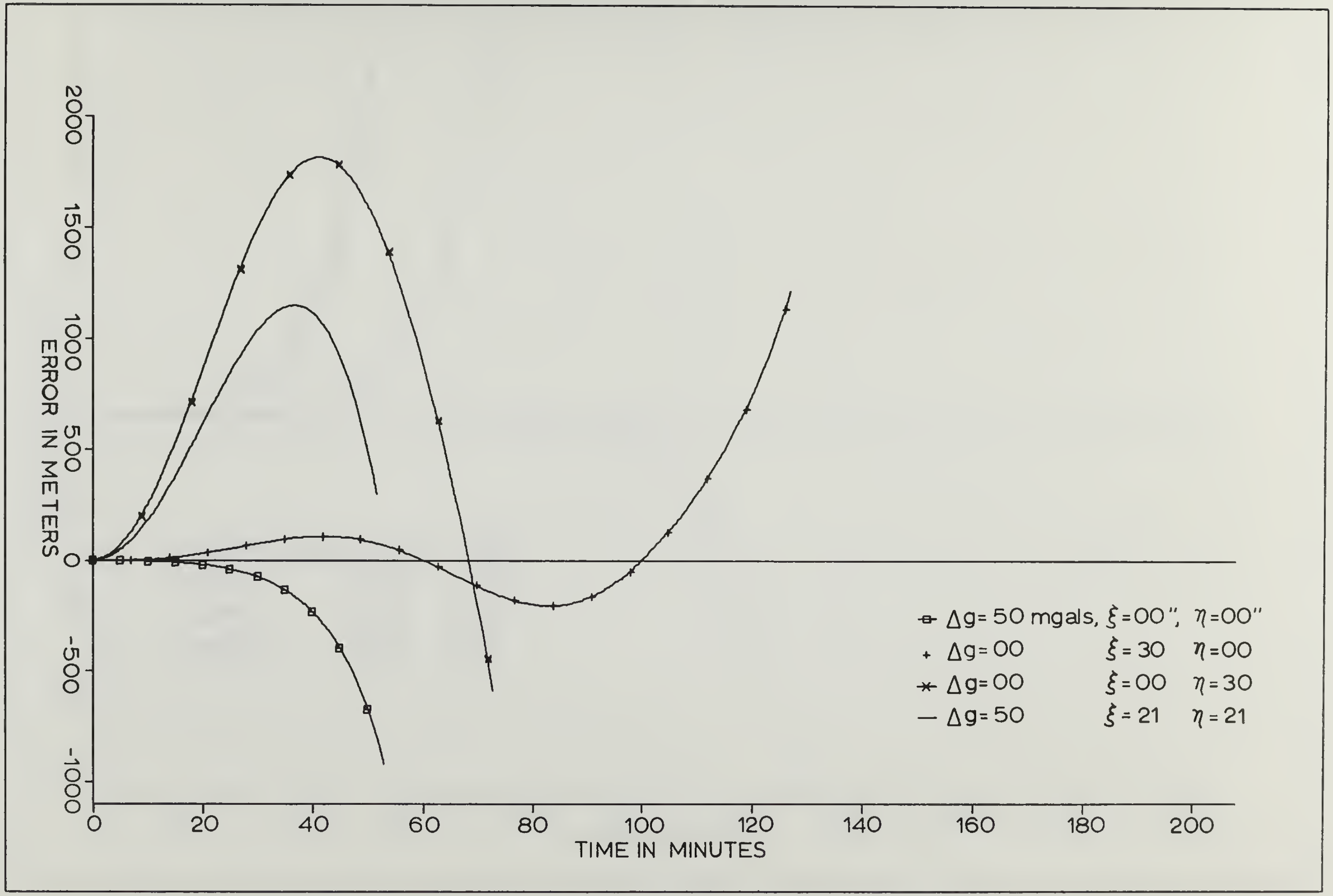


Figure 3. East geodetic error oi a stationary observer

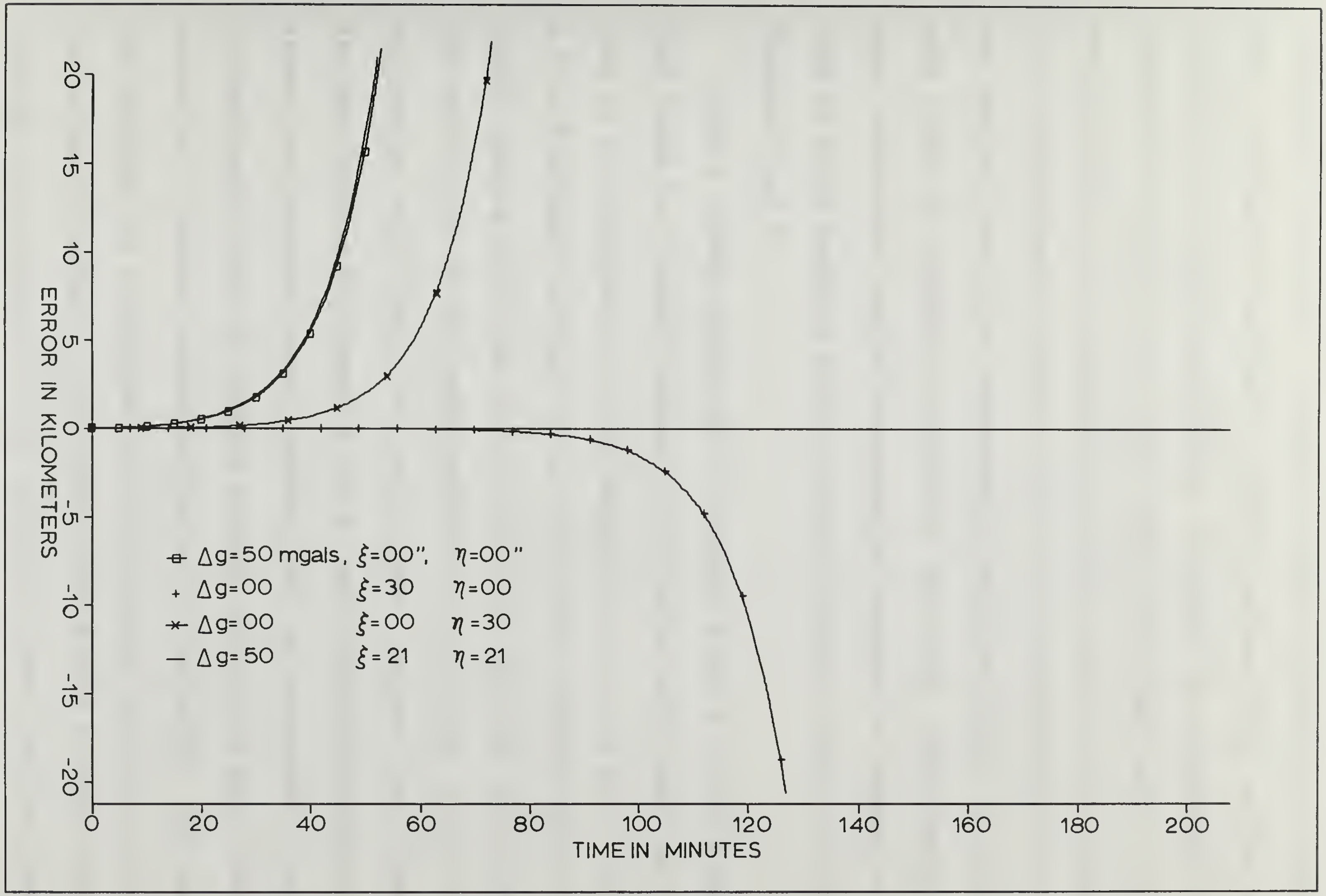


Figure 4. Height error of a stationary observer

elevation, we are able to limit the rapid propagation of height error, although we may have a fairly constant error of the uncertainty in the geoidal undulation. By using the independent determination of height, we not only reduce the height error to zero, but also reduce the component of velocity normal to the ellipsoid to an independently measured value at each computed position. In our specific case, the independently measured normal velocity is zero since the navigator is fixed to the earth. The results of using independent height measurements, depicted as before in the form of north geodetic error and east geodetic error, are shown in figures 5 and 6.

From a casual inspection of figures 5 and 6 it would appear that there is a slight damping of the north error where the deflection of the vertical is in the meridian (i.e., $\xi \neq 0$, $\eta = 0$), with a time dependent increase in the east error, whereas there is a slight damping with time of the east error when the deflection of the vertical is in the prime vertical (i.e., $\xi = 0$, $\eta \neq 0$), with an increase with time in the north error. Figure 7 is a plot of the same data as in figures 5 and 6, but with north geodetic error versus east geodetic error, showing that the navigator's position is oscillating with the Schuler period, and at the same time rotating as a Foucault pendulum relative to the earth. The example was computed for a latitude of forty degrees, and the position of the navigator marked at every fifth minute with a small square. Although the components of the navigation error are time dependent the magnitude of the maximum positional error with respect to the

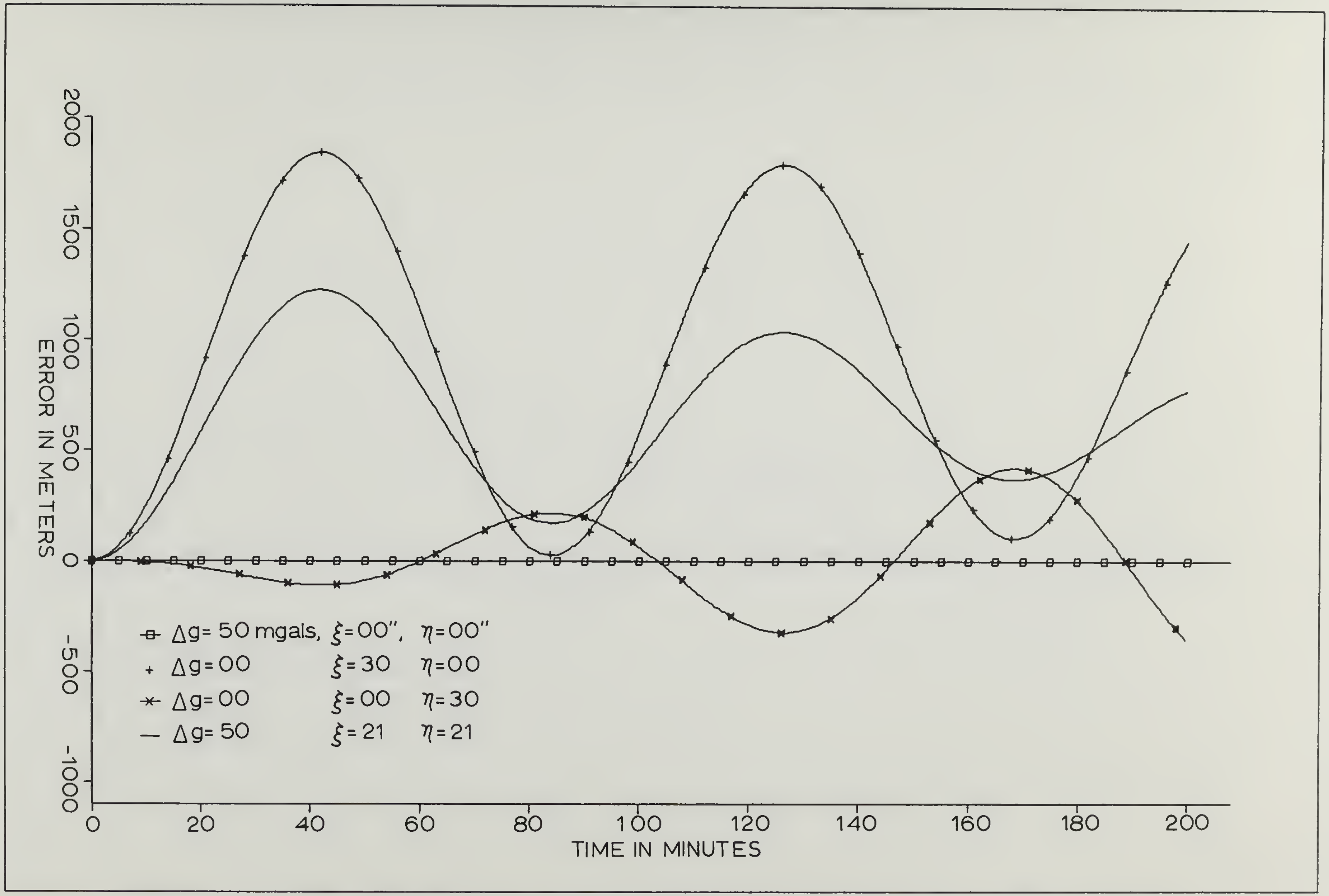


Figure 5. North geodetic error of a stationary observer at known elevation

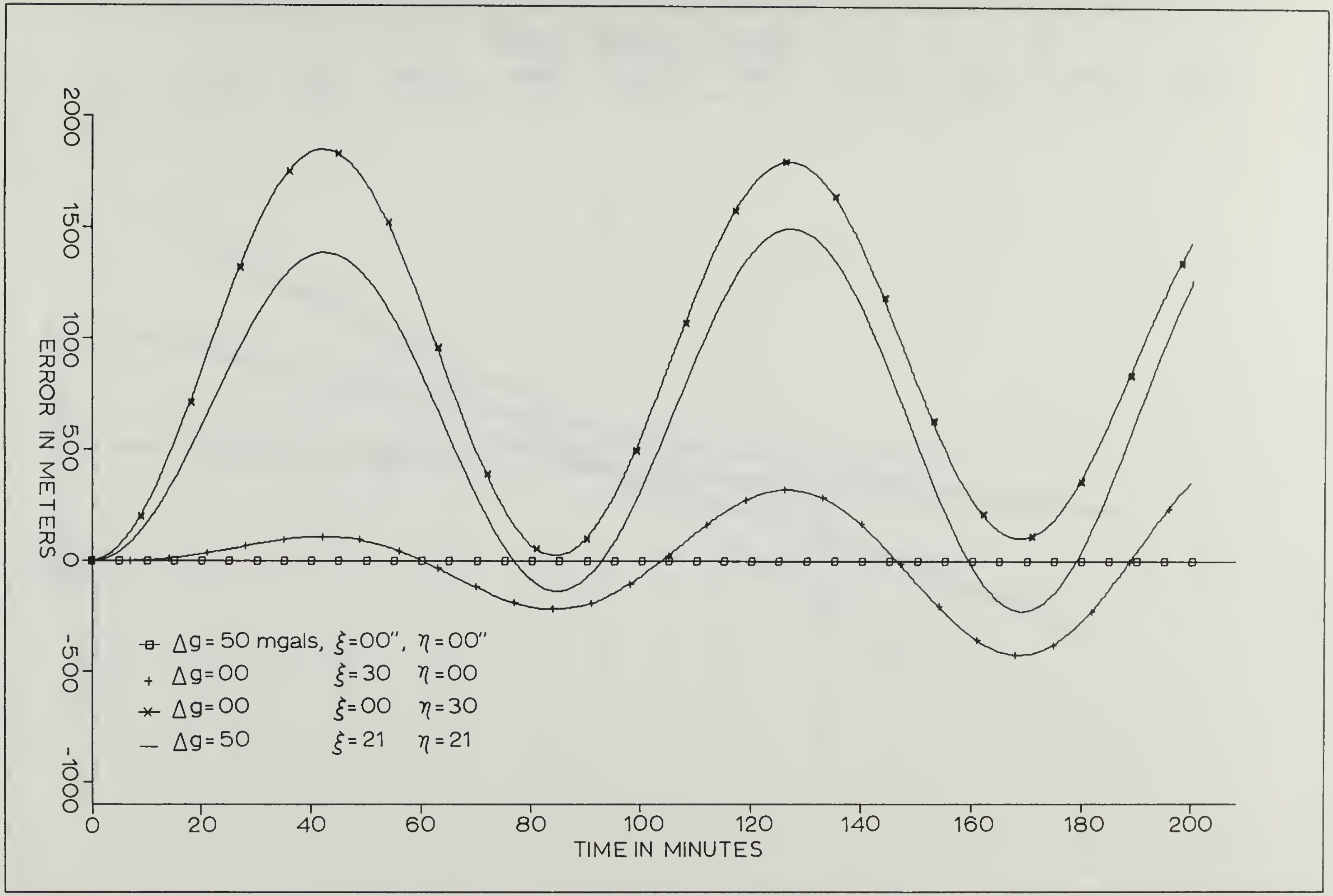


Figure 6. East geodetic error of a stationary observer at known elevation

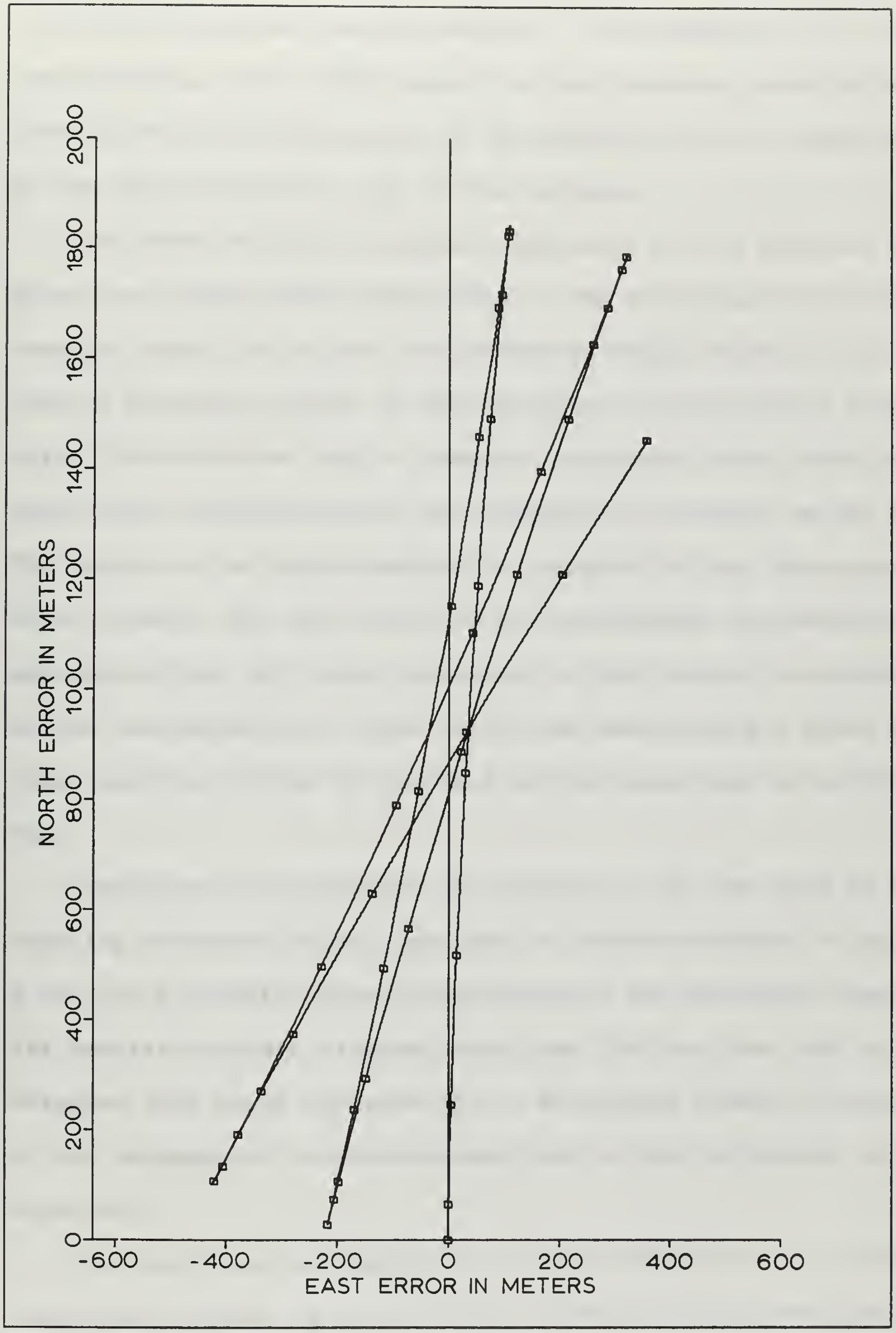


Figure 7. Horizontal geodetic error of indicated position

astronomic position remains constant. The magnitude of the maximum positional error with respect to the geodetic position varies between twice the deflection of the vertical and the square root of two times the deflection of the vertical.

The next step in conceptual complexity of the inertial navigator, over having the orientation of the reference frame fixed in inertial space, is to have the reference frame rotate at the same rate as the earth, about an axis parallel to the earth's rotation axis. This involves only a constant rotational rate about one axis, and is independent of the navigator's position on the earth. The output of the accelerometers is similar to that discussed above, except that the effect of the centripetal acceleration and mass attraction at a point maintains a fixed vector relationship to the accelerometers, rather than just maintaining a fixed magnitude, and the effects of Coriolis acceleration must be accounted for.

Equations are developed in Appendix II for the case of the rotating reference frame, analogous to those developed in Appendix I for the inertially fixed orientation of the reference frame. The results obtained by using equations II.8 are the same as those obtained from using equations I.10, within the limits of accuracy of the mathematical approximations made in the derivation of the equations.

The next step in complexity of instrumentation of inertial navigation systems is to drive the platform holding the accelerometers in such a way that the gravity vector always remains

normal to the platform. In such a system for surface navigation we can eliminate one accelerometer, the one sensing the vertical component of acceleration, and measure accelerations only in the horizon plane. This system requires that at every point where our navigator is likely to be we know the detailed shape of the geoid so that we may drive the stable platform at the meridional local vertical rate about the east-west axis, at (earth rate plus the astronomic longitude rate) times the cosine of the astronomic latitude about the north-south axis, and at (earth rate plus the astronomic longitude rate) times the sine of the astronomic latitude about the vertical axis. This will provide a continuous indication of astronomic latitude and longitude; however, since the detailed shape of the geoid is known wherever we navigate, according to the requirements of the system, the astronomic position can be converted to geodetic and the problem is solved.

A variety of things have conspired to militate against the successful operation of such a system. One of the problems is that of sheer complexity in trying to describe the geoid, to compute radii of curvature, and to drive the stable platform at the correct rates. Another point which should not be overlooked is our near-total lack of knowledge of the shape of the geoid in the ocean areas; however, for the purpose of this investigation we shall assume that we do know the slope of the geoid with respect to the adopted ellipsoid at our geodetic position.

The practical implementation of this type system has taken several different forms, all of them assuming that gravity is

normal and that the earth is an ellipsoid, and that the stable platform is "locally level." That some of these assumptions are not fulfilled is obvious, but the degree of lack of fulfillment may be illustrated by the description of the inertial navigation equipment used by the U.S.S. Skate in her transit of the Arctic Ocean as given by Dozier:

> The N6 platform subsystem mechanization to be described may be regarded as theoretically exact and error-free if it moves on a perfectly spherical surface and there are no imperfections in the components, such as gyroscopic drifts. (2)

Under the assumption of the spherical earth it was possible to divorce platform stabilization from computer operation, permitting the continued operation of the system in the event of a computer malfunction, which is certainly a very desirable feature; however, this system does introduce certain errors into the computed position because of the fact that the platform is not always level. Additionally, in other developments, as a first order approximation, height has traditionally been considered to lie along the direction the platform is being maintained perpendicular to, whether it be the geocentric radius vector, the gravitation vector, the gravity vector or the normal to the ellipsoid.

In order to get away from the various approximations made in the literature, the kinematic equations of motion for an inertial navigator on an ellipsoid of revolution are derived in Appendix III, being presented as equations III.8. Following that derivation, a treatment analogous to that in Appendixes I and II for the case of a stationary navigator in a non-normal field is carried

out, resulting in the average accelerations over a time step shown in equations III.16 and III.17.

When these equations were evaluated with the conditions of deflection of the vertical and gravity anomalies used in the two cases above, and the results doubly integrated, the errors were the same as before, indicating that the alignment of the platform axes with the normal to the ellipsoid at the navigator's position rather than at the geodetic position was a source of error of second order, and, under the conditions of non-normality of field which may reasonably be expected, may be disregarded. Although the value of position geodetic error approaches one mile at the worst condition, the difference in error between the system rotating with the earth and the system seeking the ellipsoidal normal is a maximum of 0.51 meters.

There are many variations possible in the design of the sensor orientation, but they are only minor changes from the three cases considered here. They may reasonably be expected to react in a manner similar to those investigated, so a detailed investigation of each type will not be conducted.

CHAPTER III

OPERATION OF A PERFECT INERTIAL NAVIGATION DEVICE TRANSITING A NON-NORMAL FORCE FIELD

In the previous chapter the operation of various basic types of inertial navigation devices was investigated for the restrictive case of being stationary with respect to the surface of the earth. Since each of the three basic types behaved in the same manner, within the limitations of the mathematical methods employed, we will now concentrate on the type which is representative of the majority of instruments now available, i.e., the case wherein one axis attempts to remain aligned with the normal to the reference ellipsoid.

Equations are developed in Appendix IV for the evaluation of the acceleration in latitude and longitude, accepting the fact that an independent means must be used to determine the height of the navigator if this coordinate is desired. The navigation problem undertaken using these equations considered the case of a surface vessel, which maintained a constant speed of twenty knots and a constant heading of thirty nine and one-half degrees. The fact that the vessel remained on the surface constrained the orthometric height to be zero, and the navigator's estimate of the ellipsoidal height rate to be zero.

A geoidal undulation varying from zero to one hundred meters was considered, with a complete undulation cycle occurring in ten degrees of latitude, and fourteen and two-tenths degrees of longitude. Although the height coordinate, measured from the geoid, remained zero, the height rate with respect to the ellipsoid was not zero, because of the geoidal undulations. Figures 8 through 22 show the values of the components of the deflection of the vertical, ξ and η, as a function of time. Figures 8, 9, and 10 also show the error of position, relative to the astronomic position. In this simulation, which ran over a navigation period of 39 hours, the maximum error relative to the astronomic position was 5.6 meters, occurring just after the twenty-fifth hour. This figure can be compared to the error with respect to the geodetic position of as much as 1750 meters (0.95 miles).

The results of this test can be generalized as indicating that, while assuming that navigation is taking place on an ellipsoid and that the force of gravity is everywhere normal to this surface, the inertial navigator (mechanically perfect), when initially set to proper position and velocity, will track the actual direction of gravity on the geoidal surface to a very high degree of accuracy, and will provide the angle between the gravity vector and the equatorial plane as latitude and the angle between the plane which contains the gravity vector and which is parallel to the rotational axis of the earth and the Greenwich meridional plane as the longitude. Thus, if a map of the deflection components were available with an argument of astronomic position, it

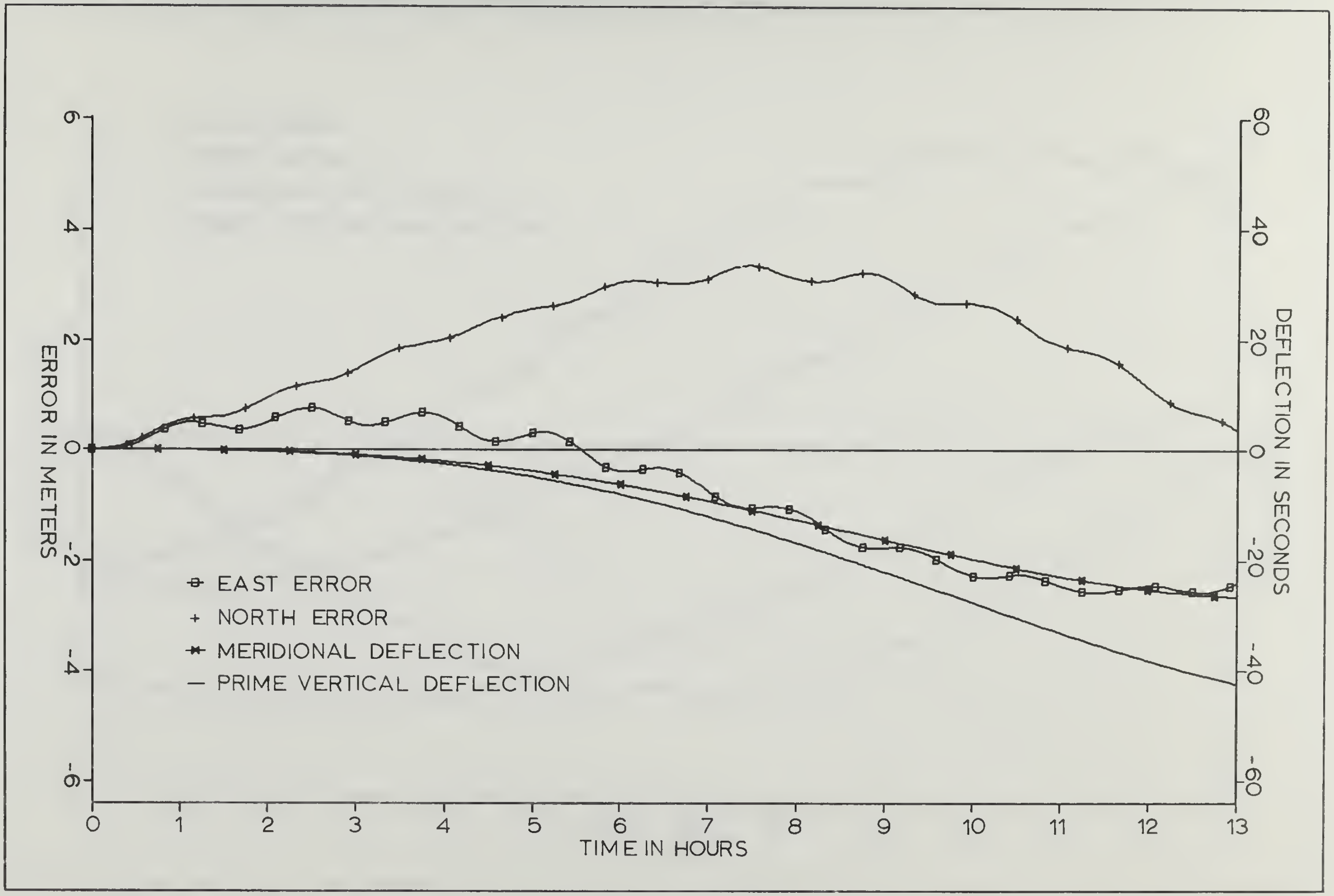


Figure 8. Error relative to astronomic position in a varying gravity field

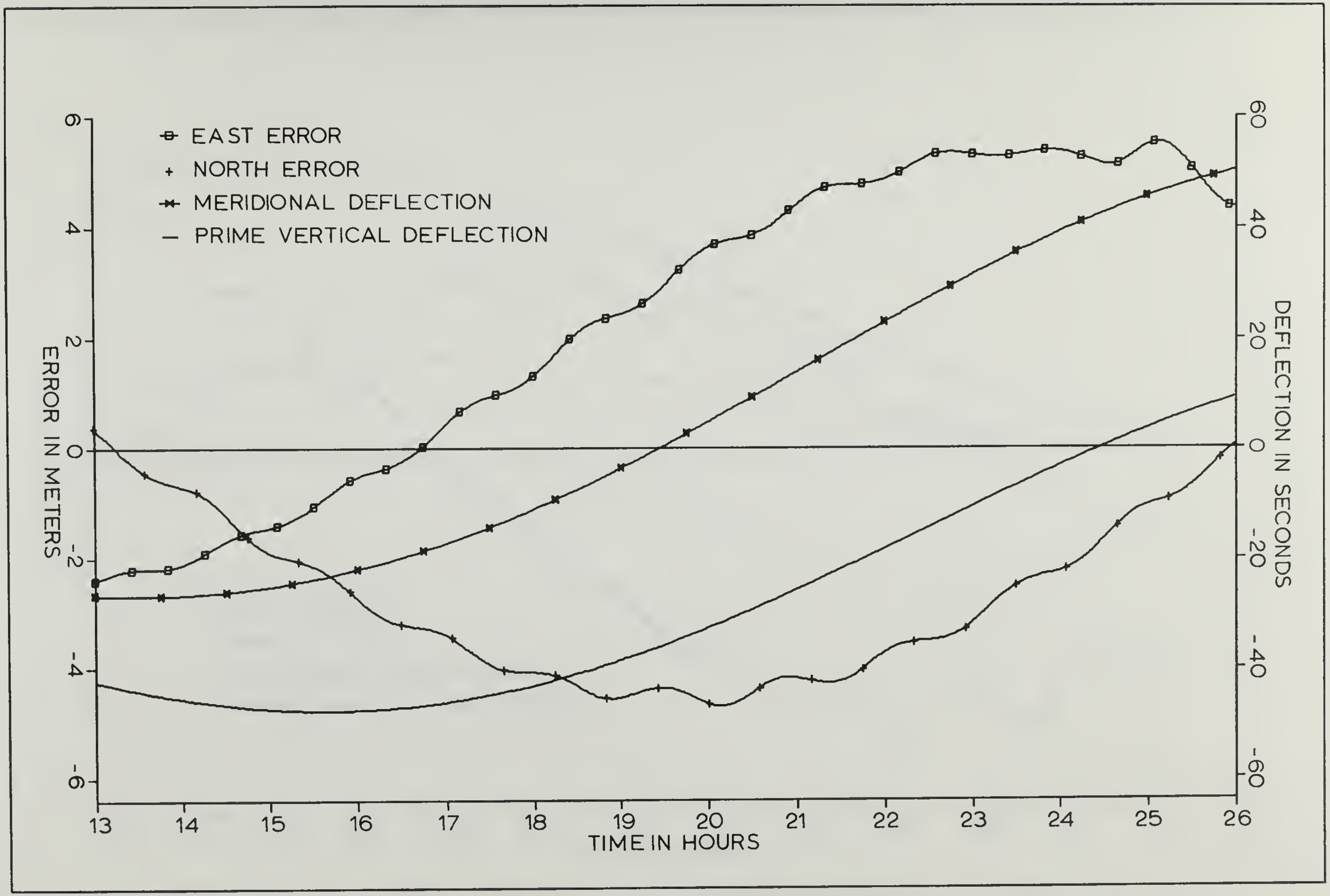


Figure 9. Error relative to astronomic position in a varying gravity field

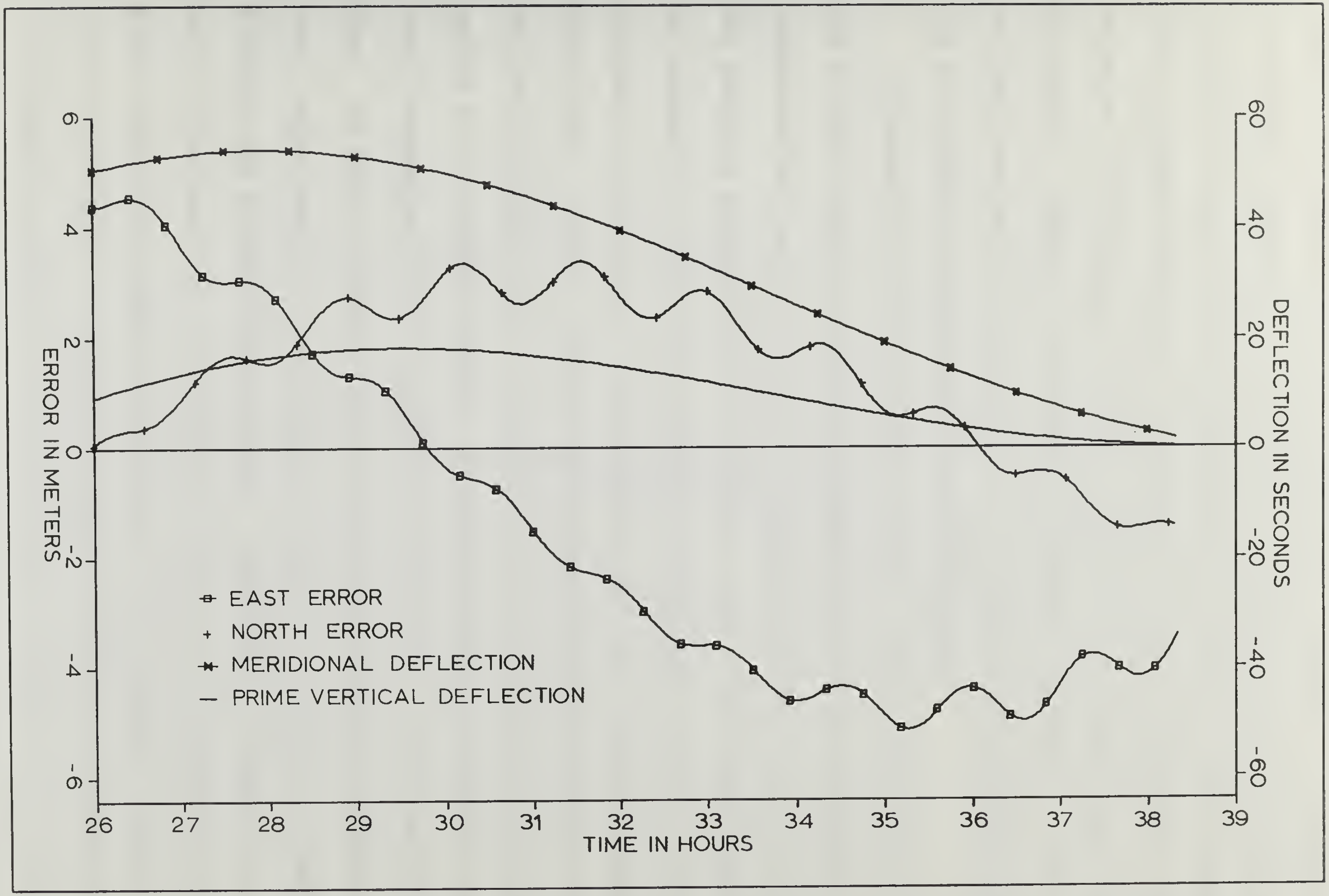


Figure 10. Error relative to astronomic position in a varying gravity field

would be a simple matter to transform the position so obtained to the geodetic position. The effect of changing the gravity anomaly from plus to minus was investigated for a period of 1000 minutes, with a maximum difference of position of eleven centimeters which occurred at 825 minutes when the value of the anomaly was $\pm$ 46.3 milligals. Although the anomaly value went as high as $\pm$ 71.6 milligals during this simulation, the position differences between the two conditions decreased after 825 minutes.

Although this study considers only a perfect inertial system and blithely ignores all the problems that have been plaguing engineers for years, we know that position errors do creep into the inertial navigator over a period of time. For this reason, it is necessary to reset the equipment periodically to the correct position and velocity. Let us assume that some external navigation system, such as the Navy Navigational Satellite System described by Russell (9), will provide errorless positioning information in an absolute system, and additionally will provide errorless information on the component of velocity in the north direction. We will use this external source of errorless information every eight hours for resetting our inertial navigator to geodetic position and actual velocity north. The results, figures 11, 12 and 13, indicate that a positional error of 3.2 kilometers (1.73 miles) has been reached, with respect to the geodetic position, within 45 minutes of the second reset. In view of the fact that the navigator attempts to track the local vertical, let us consider the error with respect to the astronomic position, but under

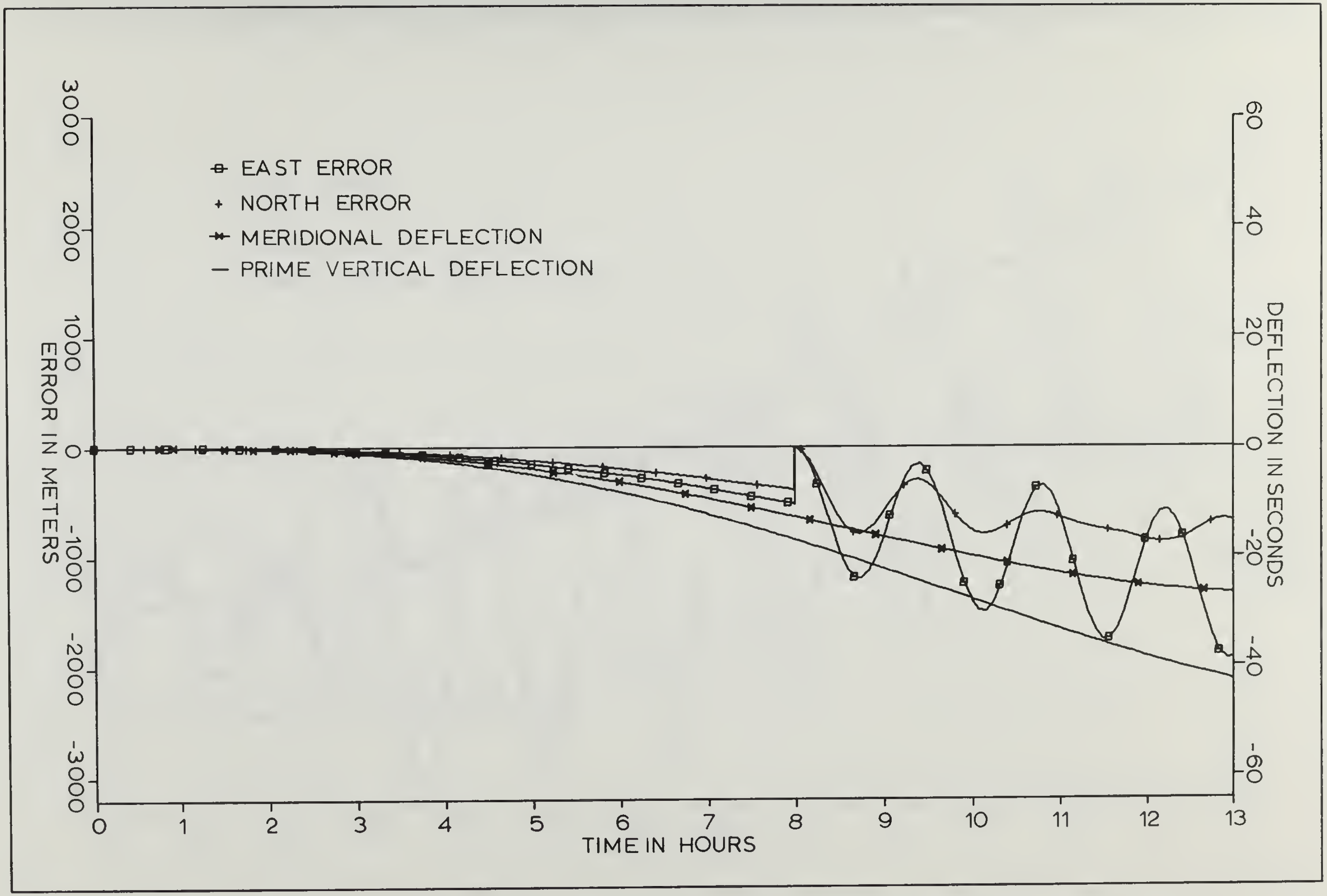


Figure 11. Error relative to geodetic position with eight-hour reset cycle

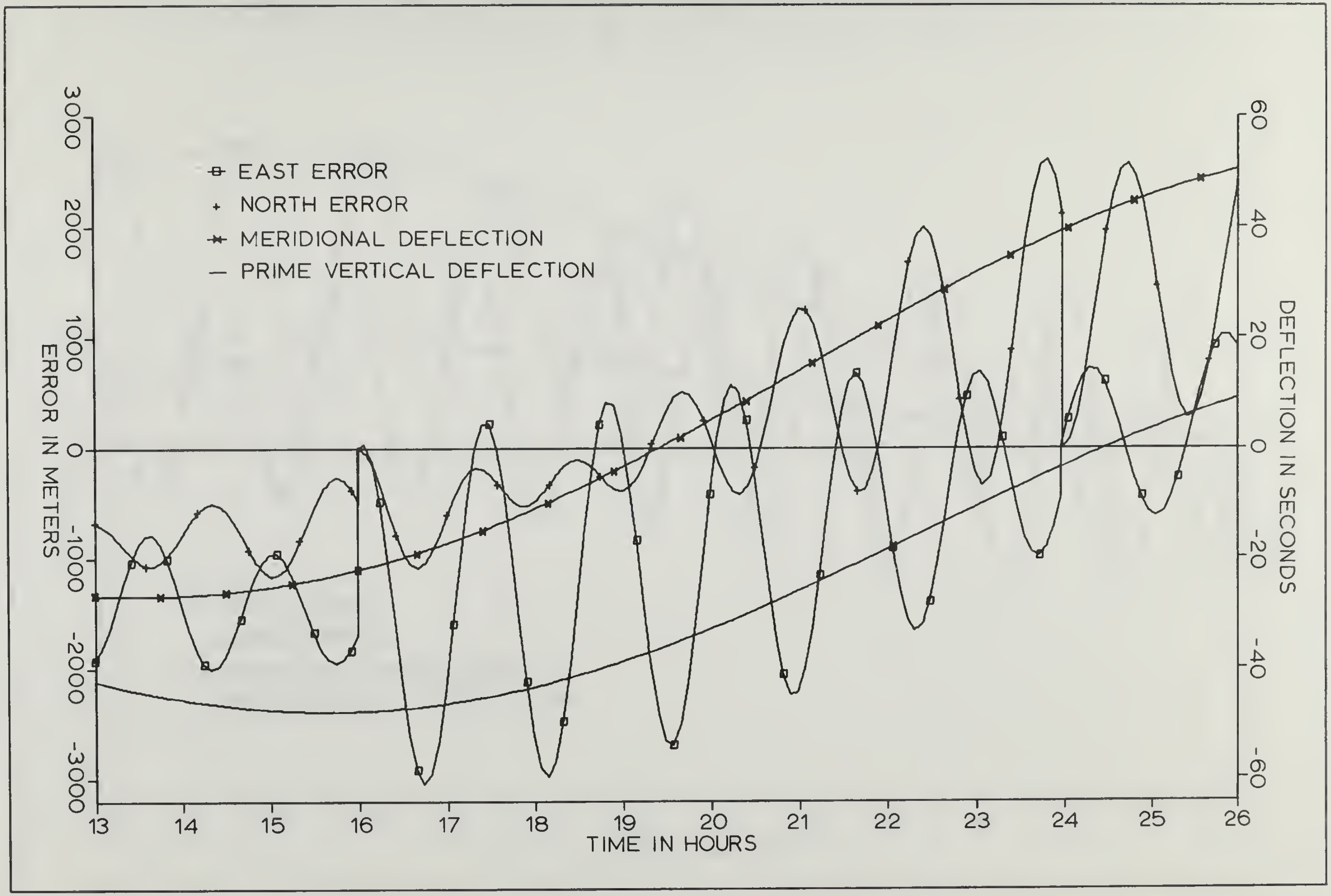


Figure 12. Error relative to geodetic position with eight-hour reset cycle

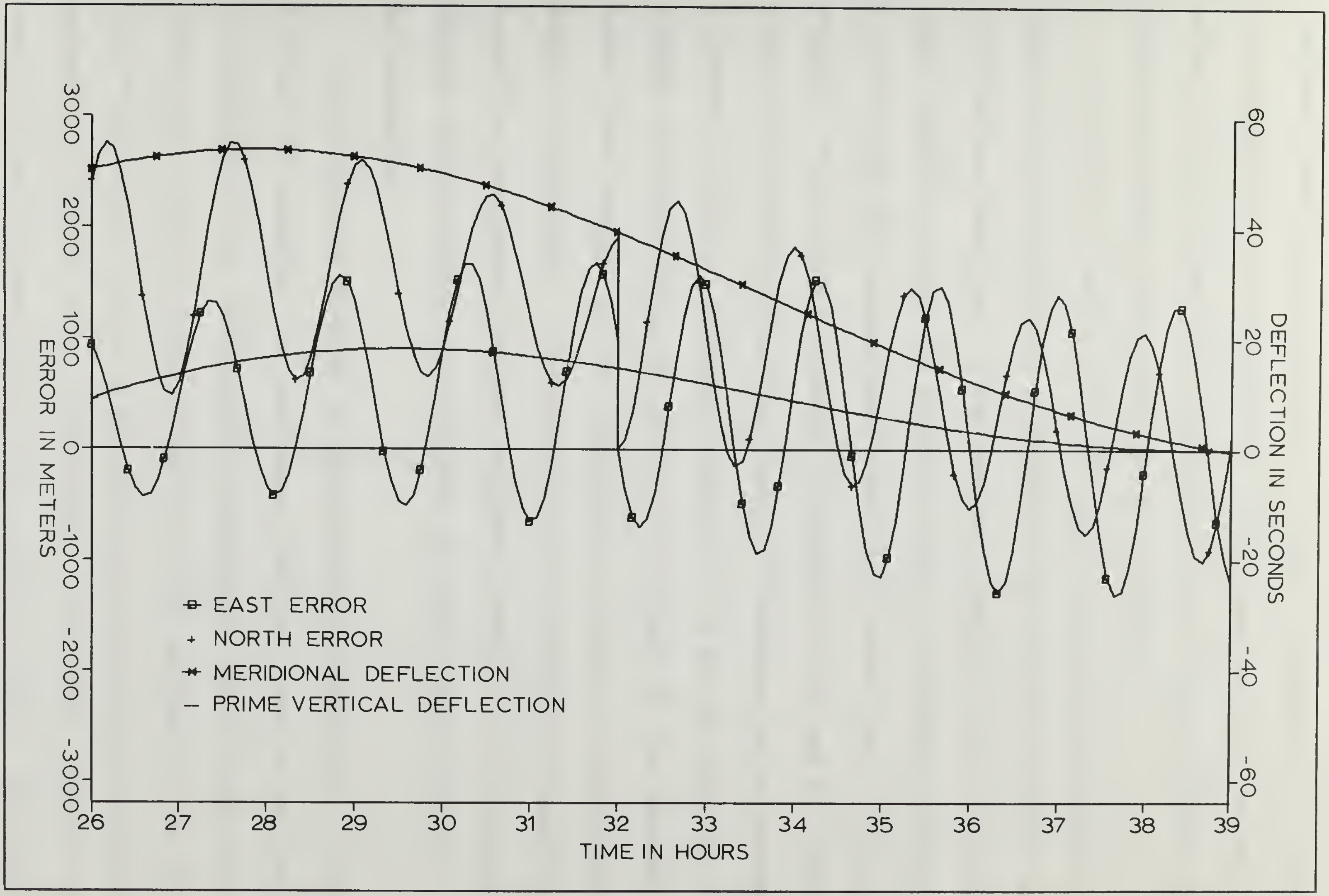


Figure 13. Error relative to geodetic position with eight-hour reset cycle

the same resetting conditions. As shown in figures 14, 15 and 16, the maximum error is reduced to 1.66 kilometers (0.89 miles), which is still considered excessive for any type of survey, or for any other purpose requiring precise navigation.

In order to exploit the fact that the inertial navigator seeks the local vertical, a simulated run was made wherein the position was reset not to the geodetic position, but to the astronomic position, while the velocity north, as obtained without error from the Navy Navigational Satellite System, was used without change. The results are shown in figures 17, 18 and 19. The reader is cautioned about the change of ordinate scale between the different figures. Although figures 17, 18 and 19 look, at first glance, as discouraging as figures 11 through 16, the maximum error, with respect to the astronomic position, is less than 52 meters (0.028 miles). This error is of a reasonable magnitude, and should be acceptable for most purposes; however, for purposes of special surveys there are instances in which this error may be excessive.

In navigating through an area where the deflection of the vertical is changing, the astronomic latitude rate of the navigator consists of two independent components. One of these is, of course, the velocity of the vessel over the surface of the ellipsoid, which, when divided by the appropriate radius of curvature, becomes the angular rate of the vessel. This item, for the latitude rate, was accounted for in the analysis above. The second item which must be considered is the time rate of change of the

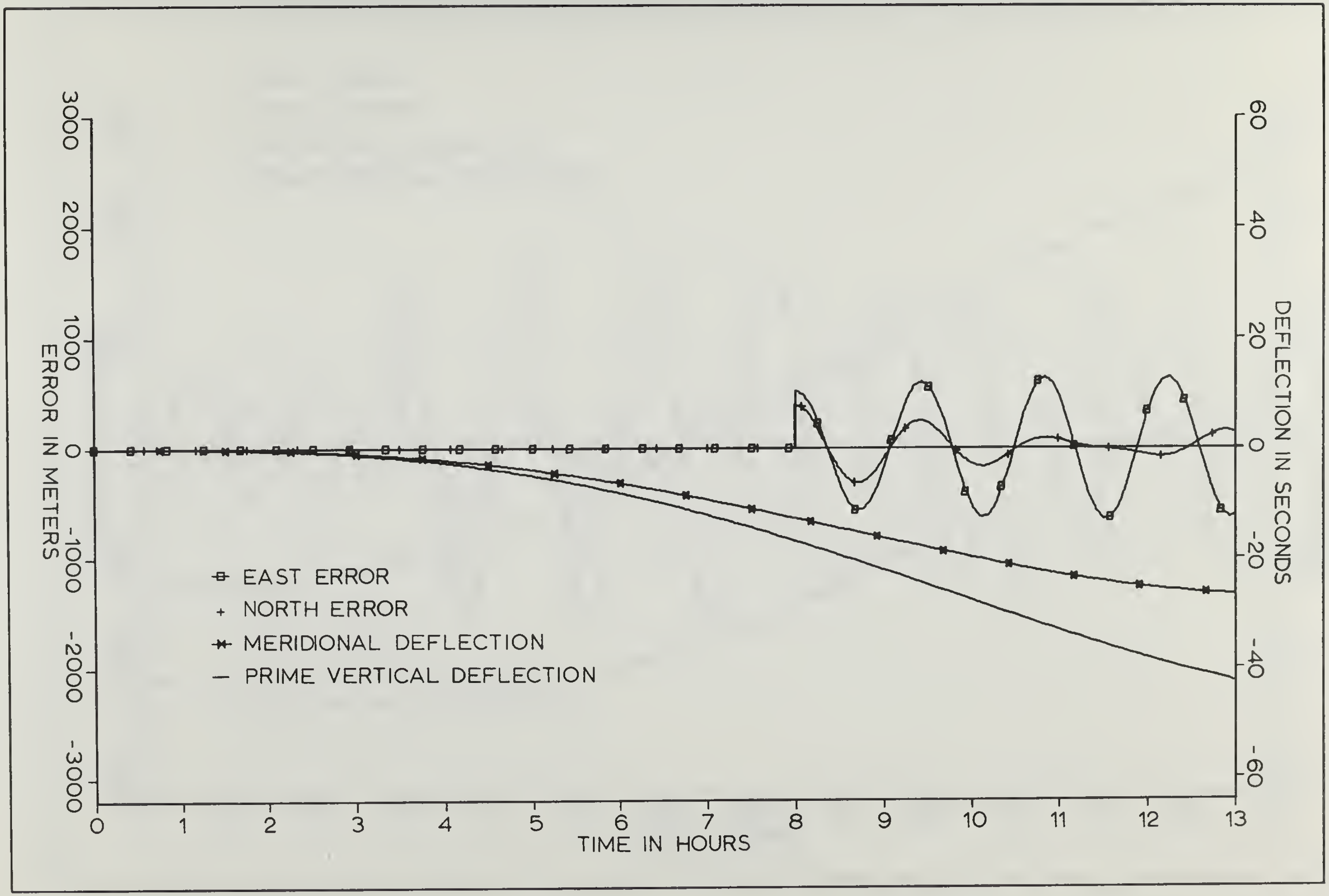


Figure 14. Error relative to astronomic position with eight-hour reset cycle

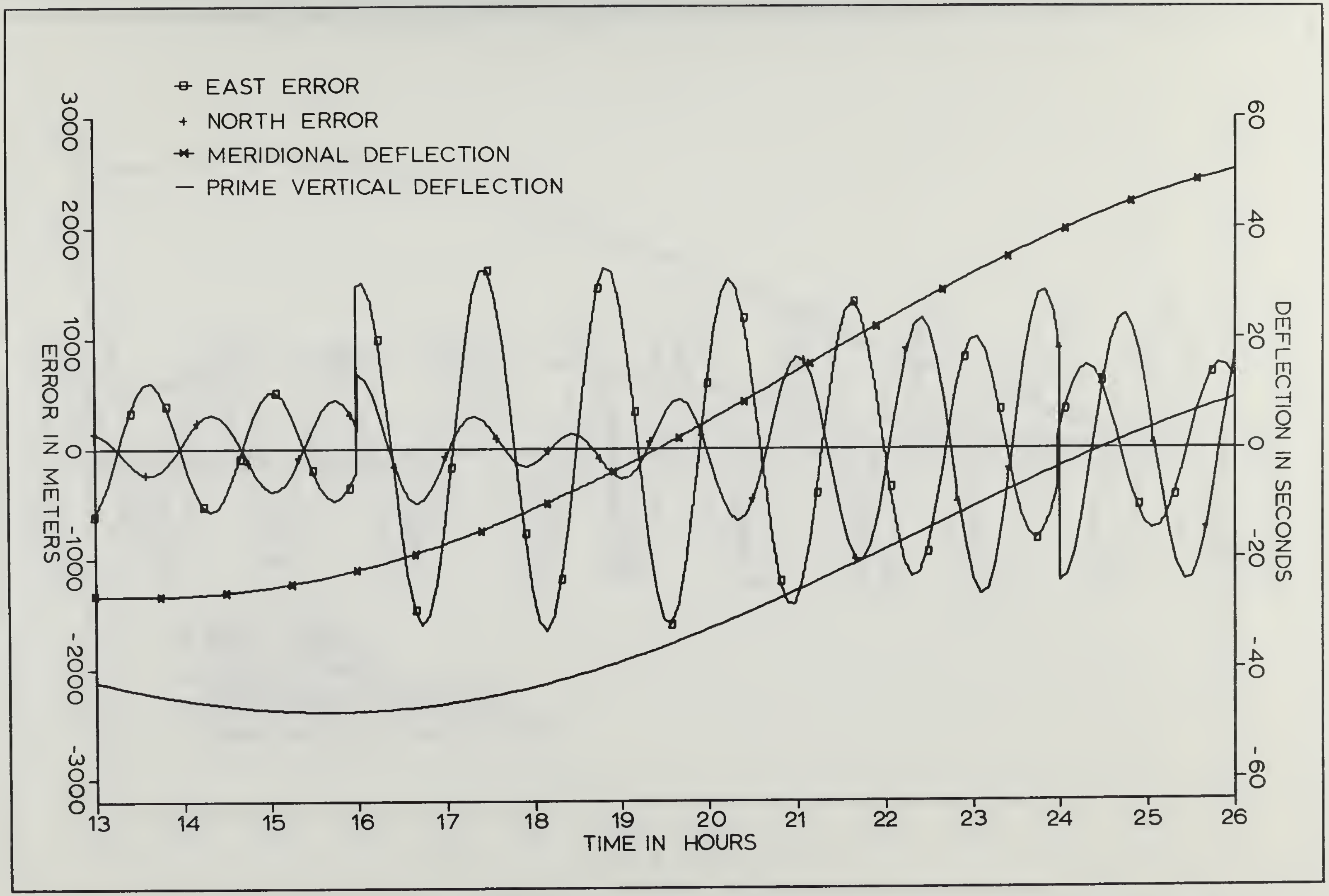
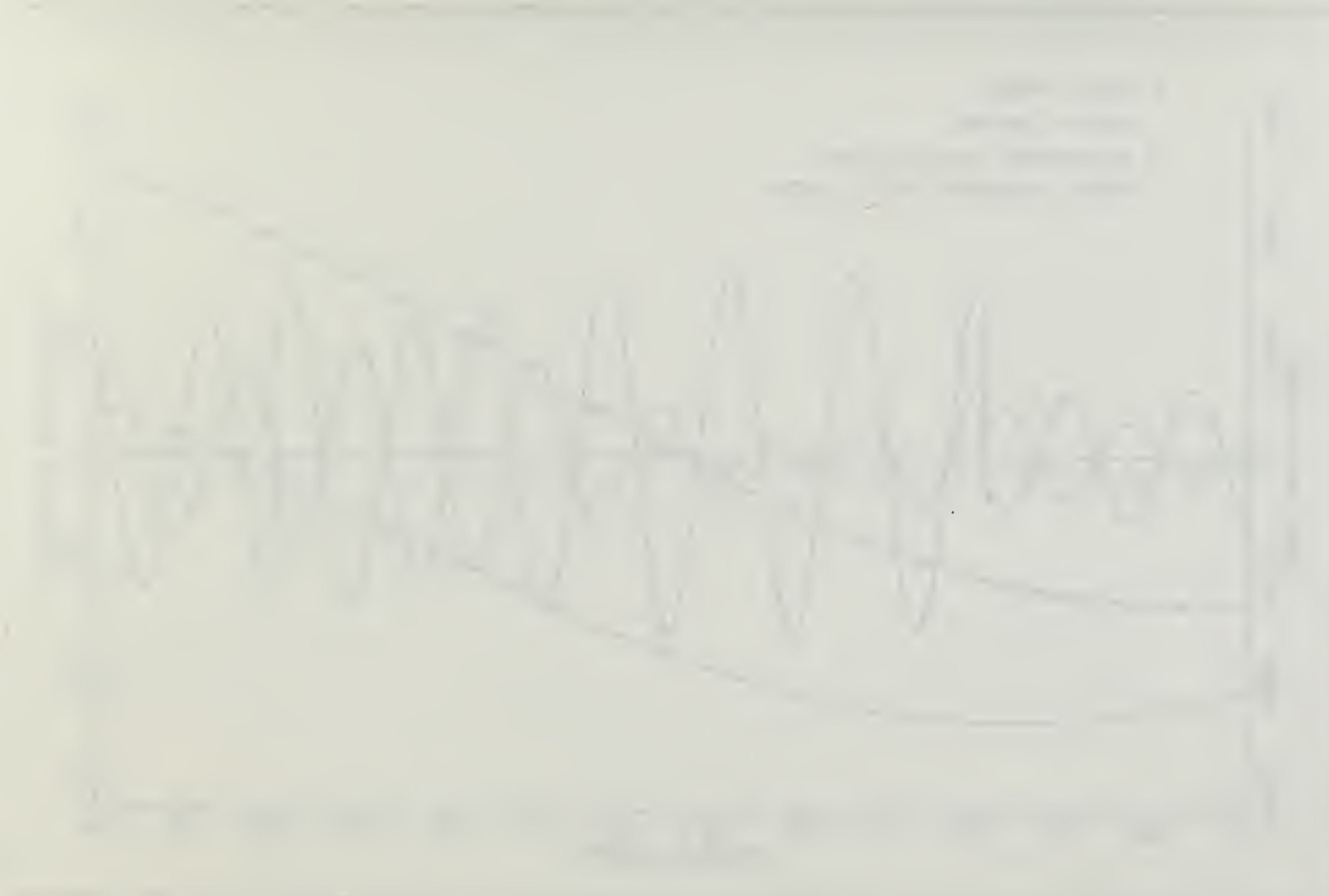


Figure 15. Error relative to astronomic position with eight-hour reset cycle



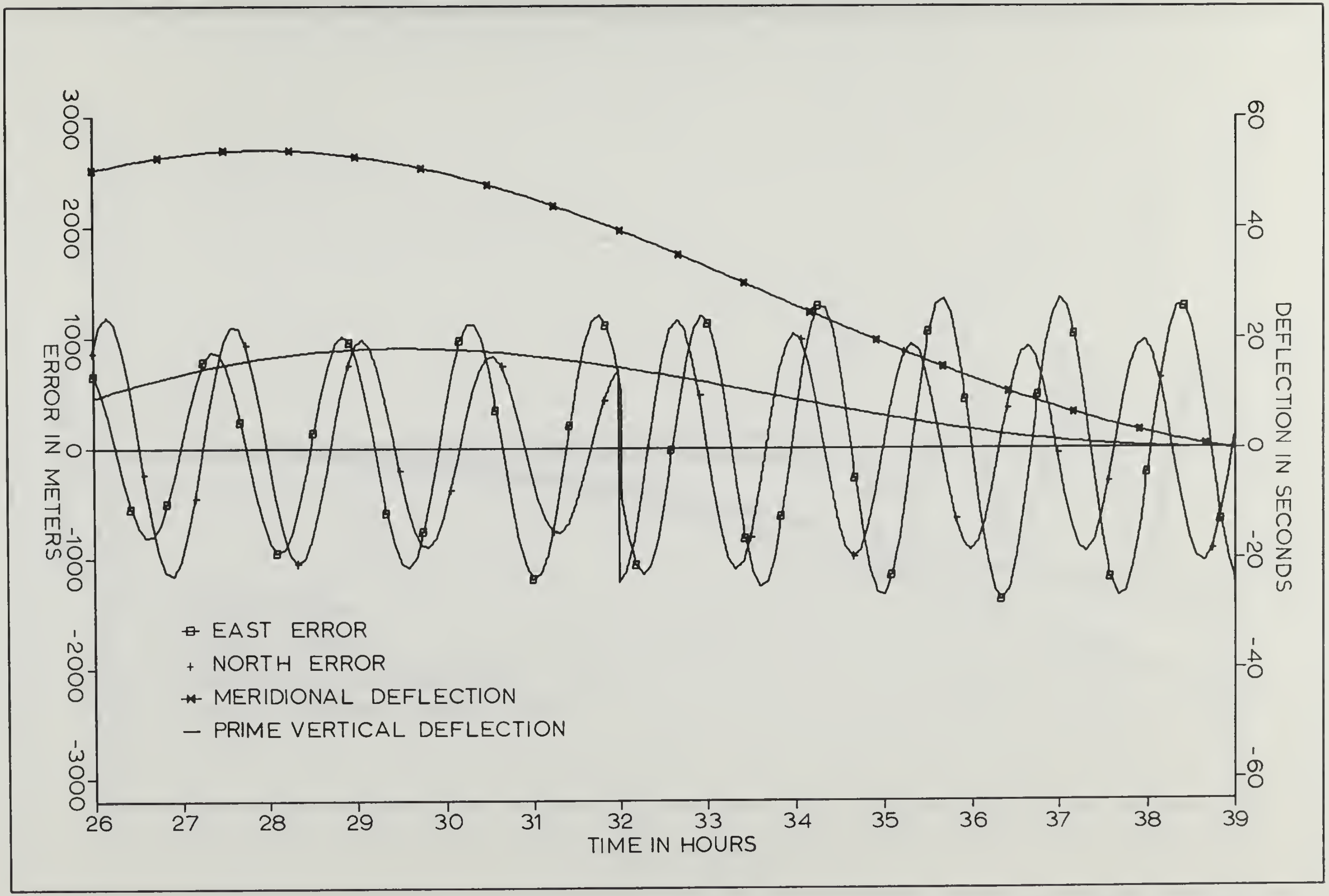


Figure 16. Error relative to astronomic position with eight-hour reset cycle

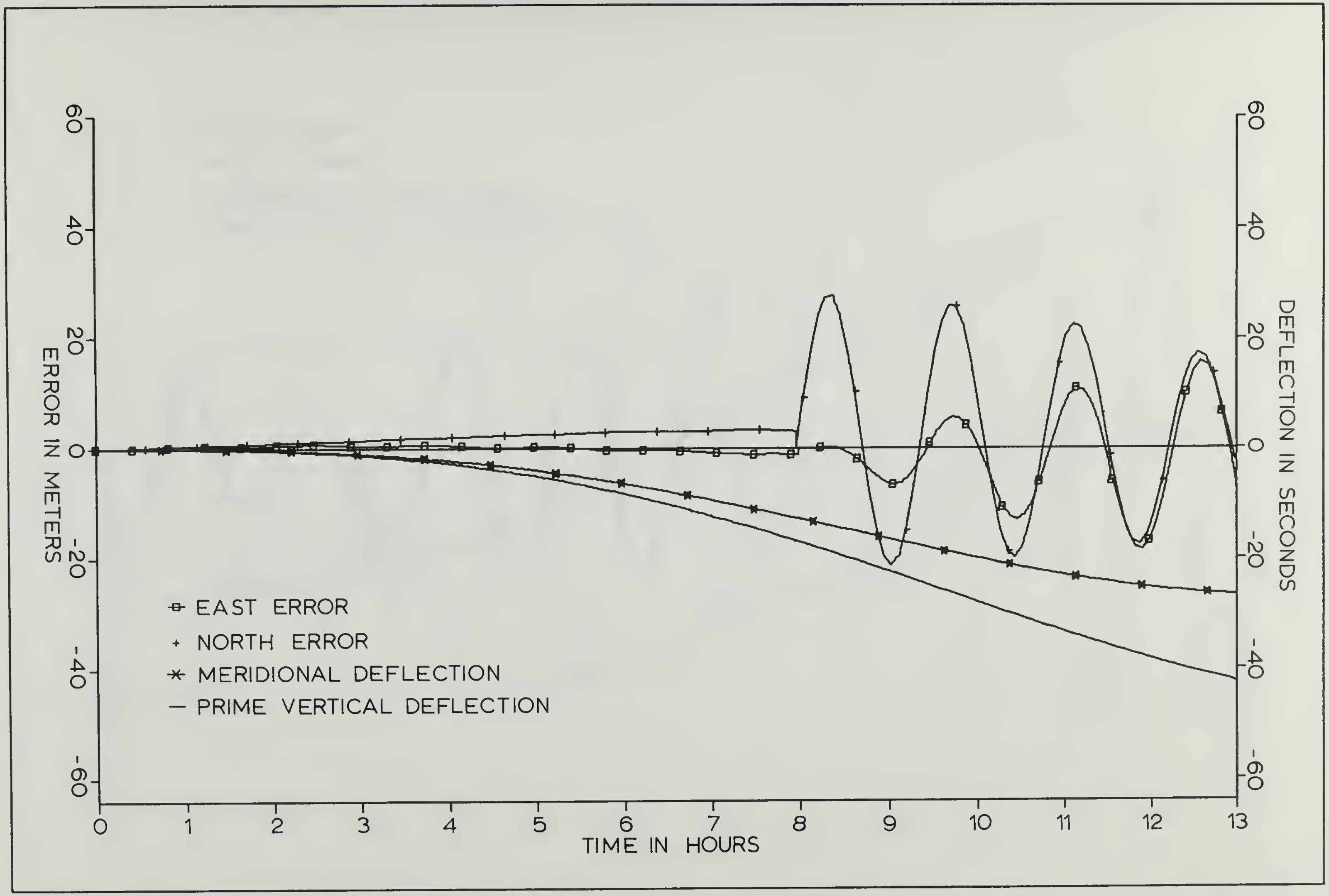


Figure 17. Astronomic error with position and velocity north reset every eight hours when rate of change of deflection is ignored

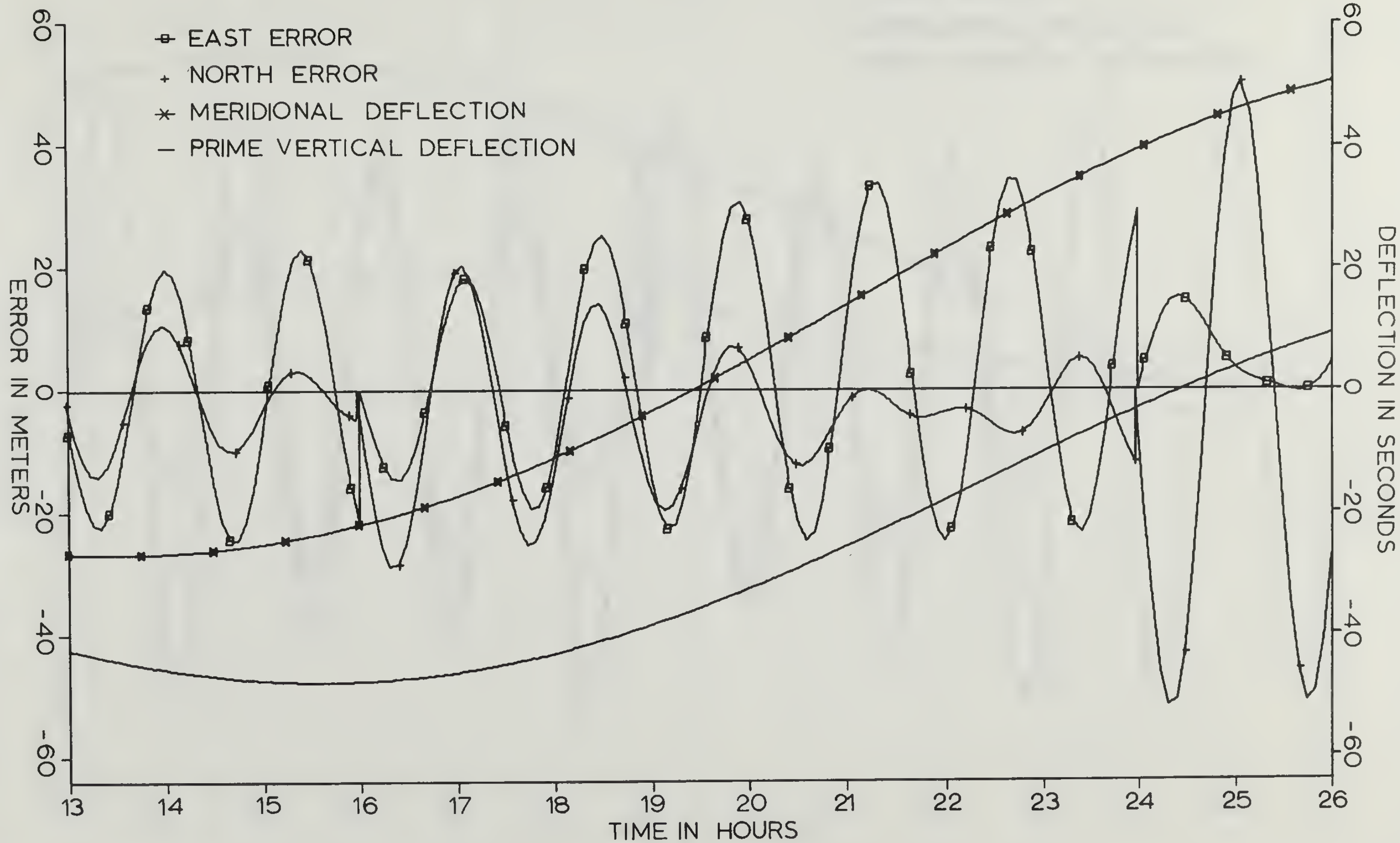


Figure 18. Astronomic error with position and velocity north reset every eight hours when rate of change of deflection is ignored

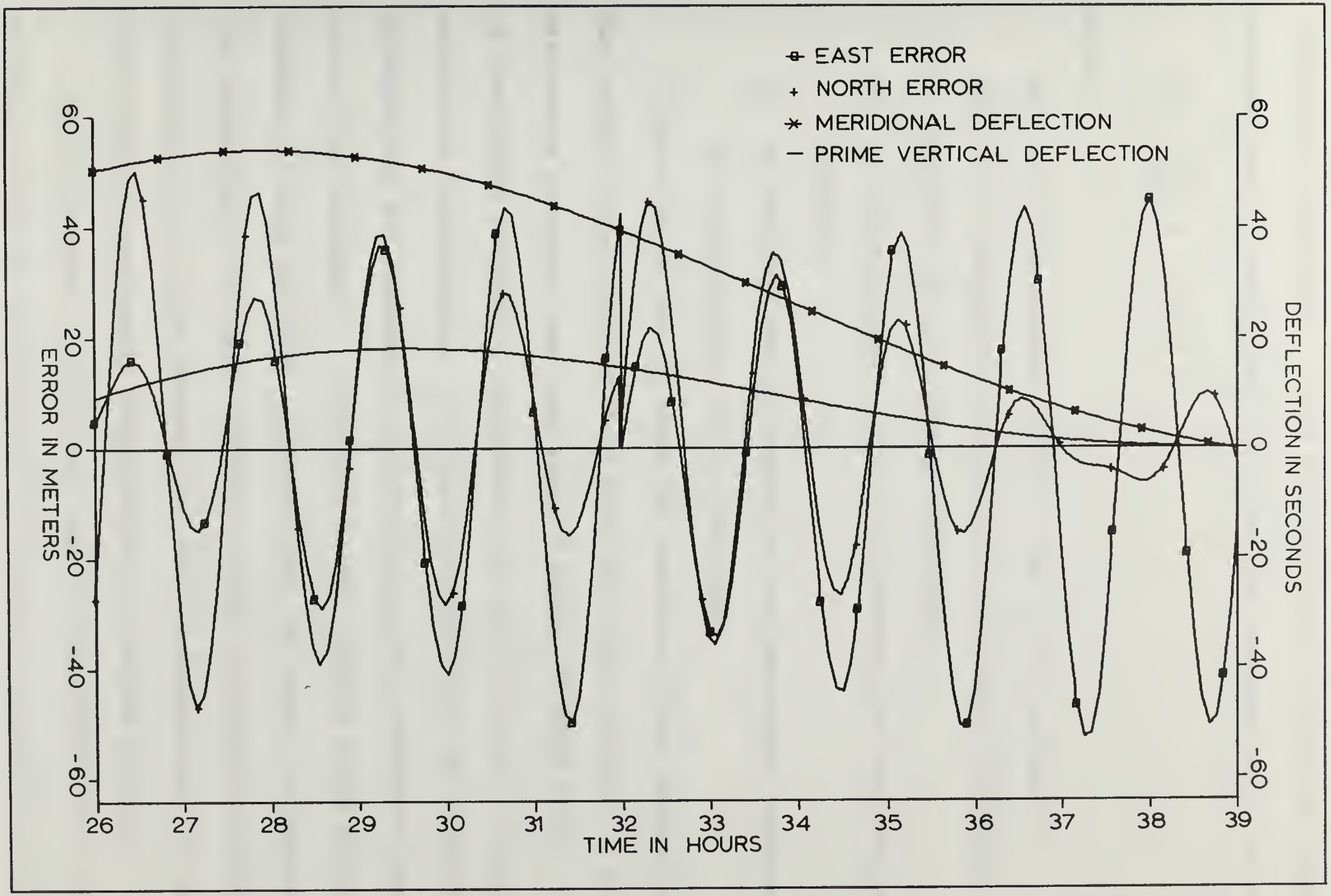
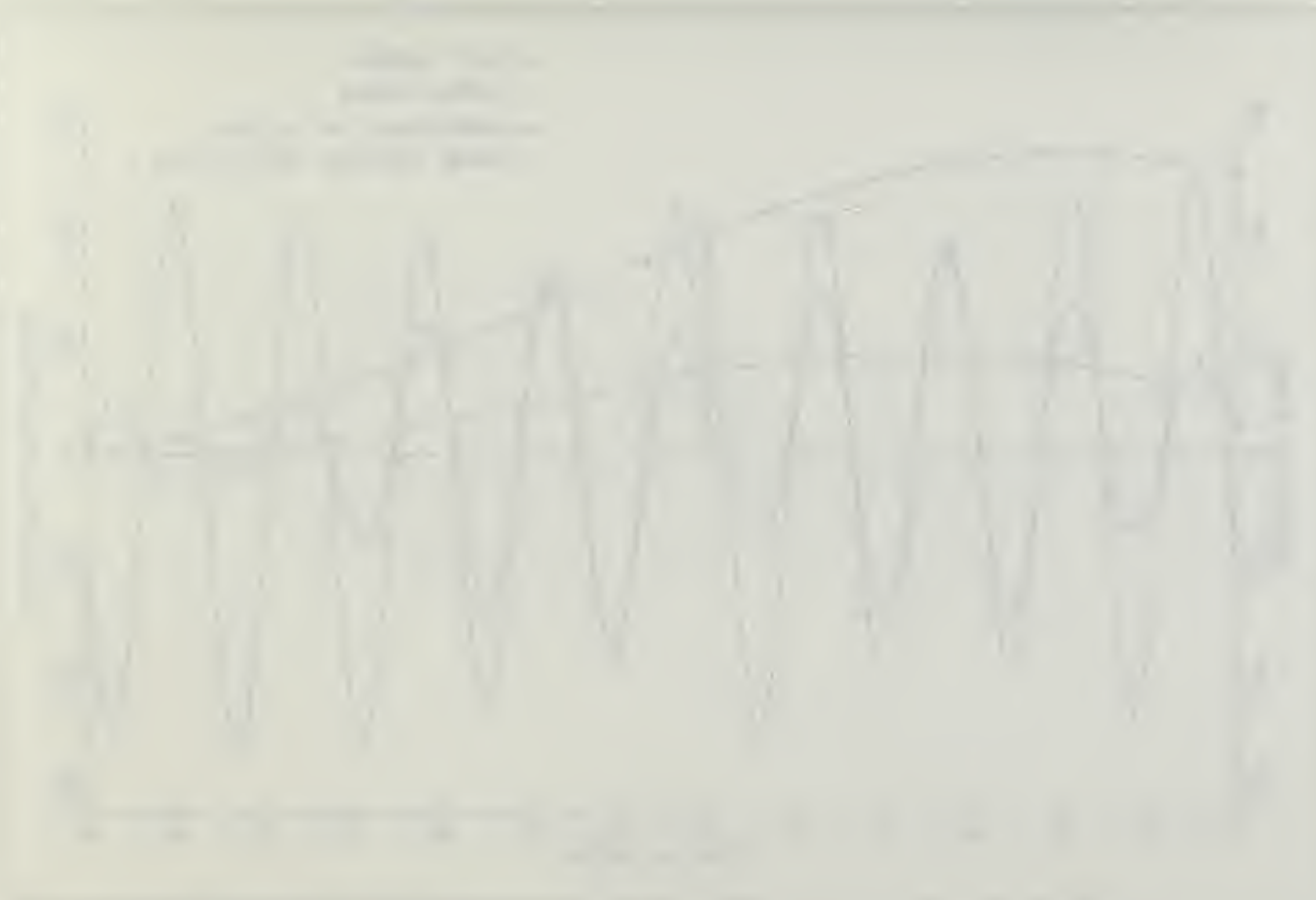


Figure 19. Astronomic error with position and velocity north reset every eight hours when rate of change of deflection is ignored



deflection components. In the present case, in which we consider external data affecting only position and latitude rate, we set

$$\dot{\varphi}_N' = \frac{V_n}{M_N} + \dot{\xi} = \dot{\varphi}_N + \dot{\xi}$$

where

$\dot{\varphi}_N'$ is the astronomic latitude rate of the navigator

V_n is the component of velocity in the north direction as provided by an external data source

M_N is the meridional radius of curvature for the navigator's latitude

$\dot{\xi}$ is the time rate of change of the meridional component of the deflection of the vertical.

Figures 20, 21 and 22 show the results of this simulation. The maximum error introduced in this case, with respect to the astronomic position, was less than 12 meters (0.0065 miles), which is considered completely acceptable for even the most stringent navigation requirements. It should be noted that the geoidal undulation was not considered in determining the astronomic latitude rate of the navigator. Presumably better results could have been obtained if this had been accounted for, but since the magnitude of the undulation is so small compared to the ellipsoidal radius of curvature, which ranges between 6.3 and 6.4 megameters, it was considered an unnecessary complication to include it here.

The environment in this test case is considered to be quite severe, in that an undulation of 100 meters is periodic in less than 850 miles. For a specific case of navigation, considering

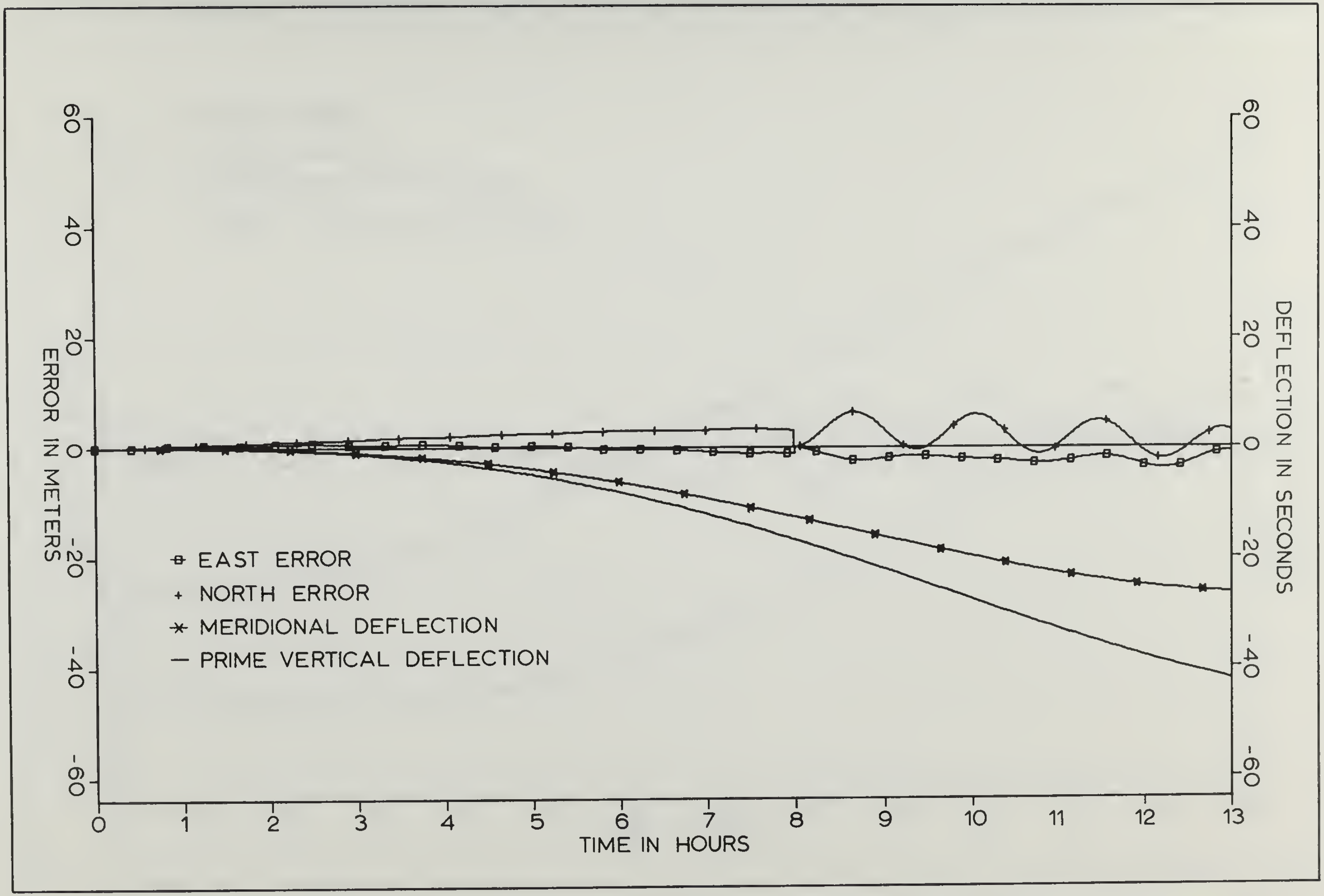


Figure 20. Astronomic error with position and velocity north reset every eight hours when rate of change of deflection is considered

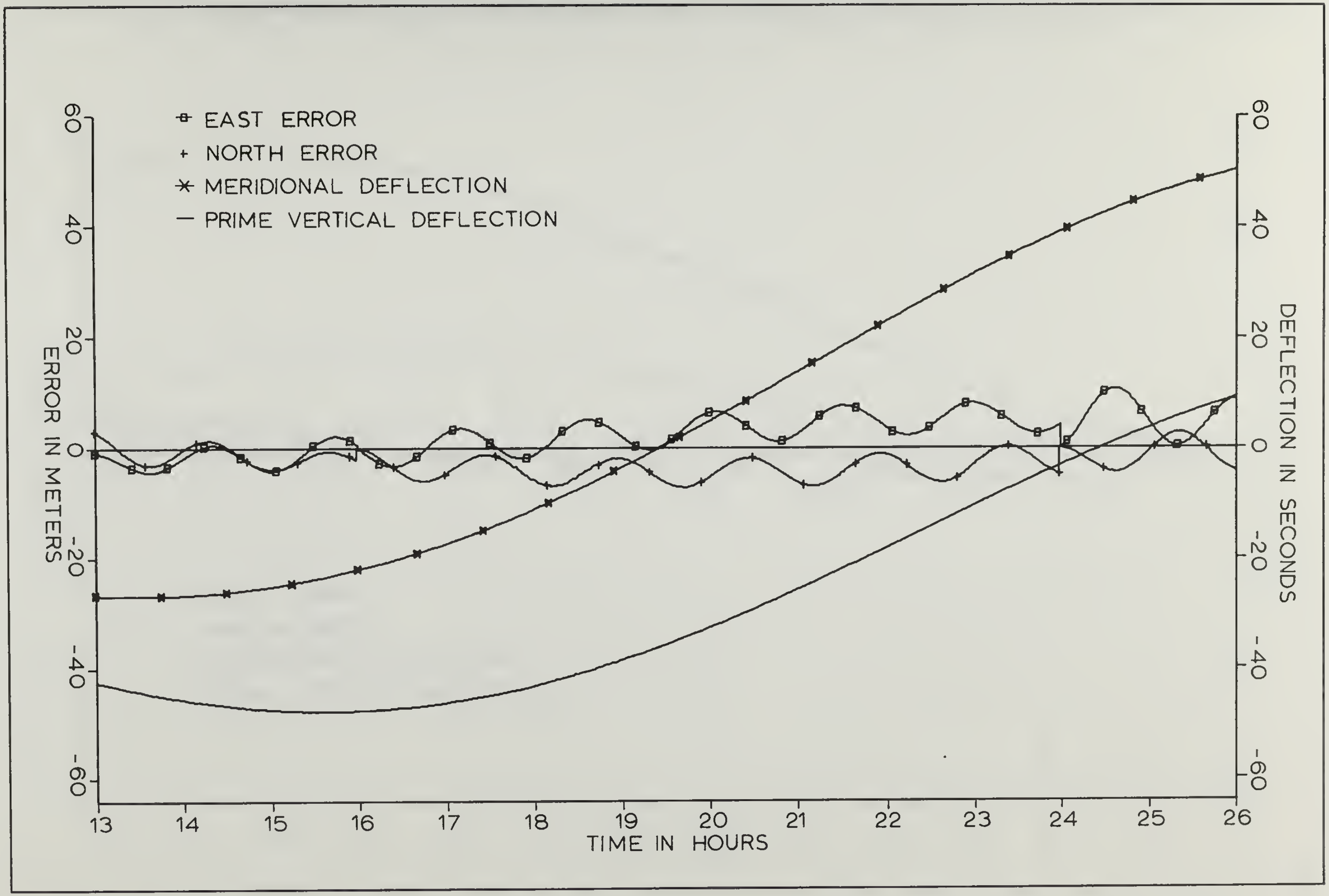


Figure 21. Astronomic error with position and velocity north reset every eight hours when rate of change of deflection is considered

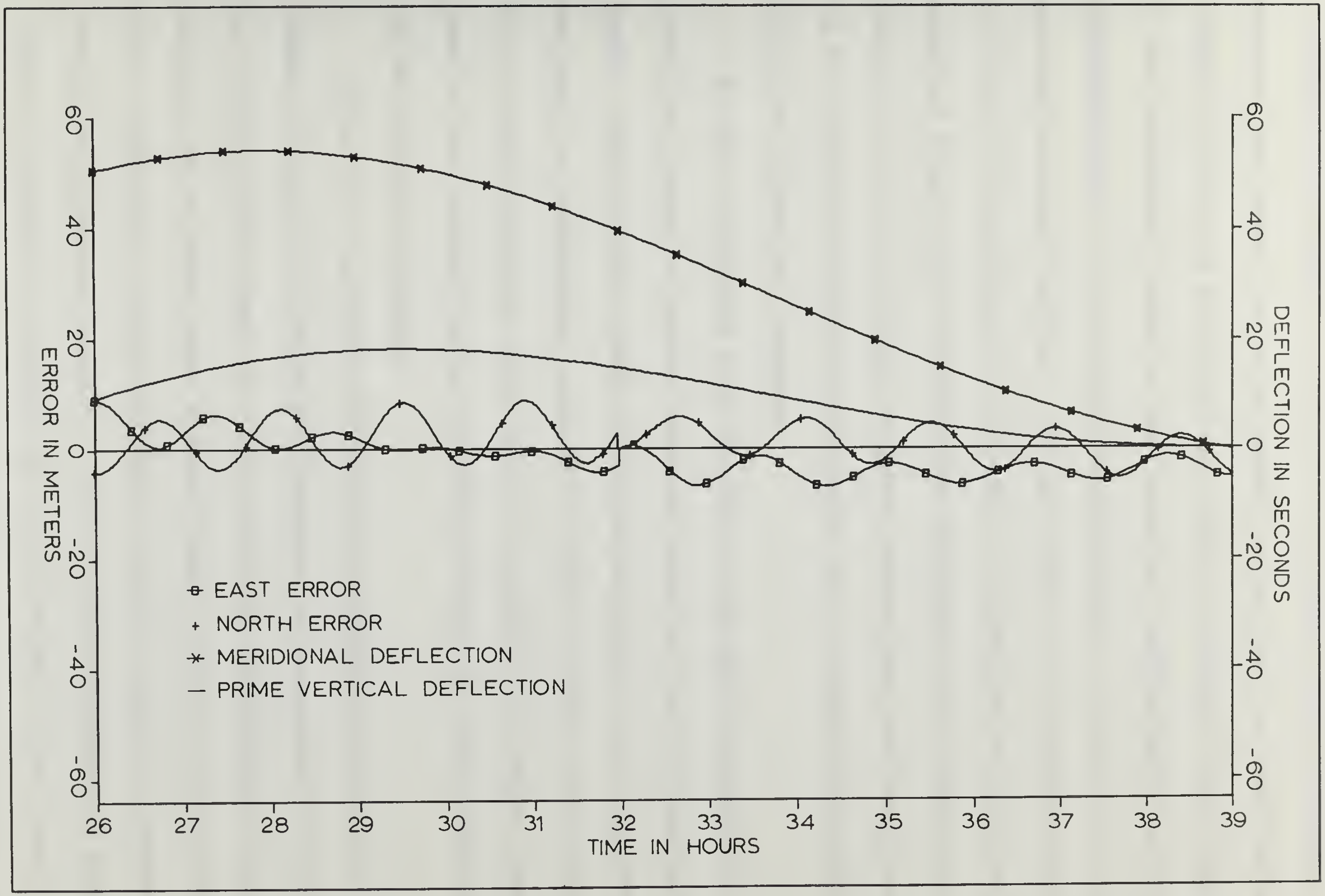


Figure 22. Astronomic error with position and velocity north reset every eight hours when rate of change of deflection is considered

the slope of the geoid in the area of interest and the maximum speed of the vessel to be navigated, it may well be that the inclusion of $\dot{\xi}$ is not warranted and that the velocity north divided by the meridional radius of curvature is a sufficiently accurate approximation of the astronomic latitude rate to provide the navigational accuracy required. In any event, it is considered that if $\dot{\xi}$ is to be considered, it should be determined for the geodetic position from a plot of deflection components, and entered manually into the navigational computer rather than utilizing storage in the computer to describe the geoid and to compute the deflection rate within the computer.

We have considered only the use of the Navy Navigational Satellite System to provide external data for the inertial navigator. Other systems, such as bottom navigation, could provide not only the information considered above, but also the velocity in the east direction, which would be handled in a manner analogous to the velocity north to further reduce the error after reset of the inertial navigator.

In order to navigate and obtain the geodetic position at all times, it would be necessary to compute the components of gravity in the horizontal plane at each step of the navigational position updating which takes place several times a second in most navigators. This would require extensive computer time and increase the storage requirements greatly. By the simple expedient of resetting the navigator's position to the astronomic rather than to the geodetic position, and by inserting the astronomic latitude and

longitude rates (which include the rates of change of the deflection components), we are able to accomplish precise navigation without introducing an increase in the time required per iteration, or increasing the size of the navigational computer. In order to use the information thus produced by the navigator it is necessary to consider the difference between the local vertical and the ellipsoidal normal, but this can be done by graphical means or in another computer, and done only when required, rather than at each step of the iterative navigational problem.

We have considered only the case where the externally supplied information is errorless. Considering the pendulum type effect of positional errors, we can conclude that an error in the position to which the inertial navigator is reset, regardless of cause, will cause the half-amplitude of the error shown in the figures (17 through 22) to be increased by the amount of the error. Since the Foucault rotation must be considered, the half-amplitude of the individual components cannot be increased according to the north error or east error, as appropriate, but the magnitude of the horizontal error must be added to the magnitude of the indicated error. The positional error at reset can be caused by either incorrect data from the external navigation device, or by errors in the determination of the deflection of the vertical. The error which is indicated in figures 17 through 22 is the result of velocity errors at the time of reset, primarily, and the result of the difference between the mathematical model we have used for the earth and the true earth shape secondarily. If we

consider the maximum error shown in figure 21 (12 meters) to be caused entirely by a velocity error at reset, it represents 0.03 knots error. An error of 0.1 knots (0.0515 meters per second) would result in a positional error of 41.5 meters half-amplitude, with a Schuler period.

CHAPTER IV

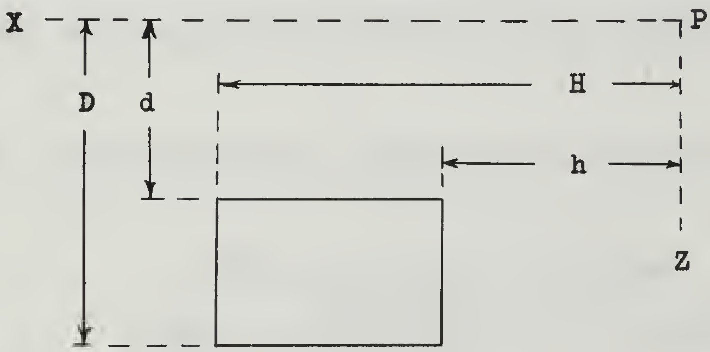
MAGNITUDE OF DEFLECTION VALUES WHICH MIGHT REASONABLY BE ENCOUNTERED AT SEA

Now that we have seen the effect of the deflection of the vertical on the operation of a perfect inertial navigator, we may logically turn to the question of the magnitude of deflection of the vertical values which might reasonably be encountered at sea. If a complete gravity survey of the world were available it would be possible to map the deflections for any area of interest to see whether or not they were large enough to be concerned about, and to make compensation for when resetting the inertial navigator. In the absence of an adequate world-wide survey, it is necessary to turn to other means to determine whether or not we need an accurate determination of the deflection of the vertical in the ocean areas.

Rice (8), in his study of the area 300 kilometers by 1800 kilometers reaching from Cape Kennedy, Florida, to the southwest corner of Puerto Rico, which is largely an ocean area, obtained deflections as great as 56.7". This area can be considered as extreme, since on one side of it there are extensive banks which are only a few fathoms deep, with the other side plunging rapidly to several thousand fathoms, and we would be inclined to avoid it

if we wanted to use inertial navigation without compensating for the deflection of the vertical. Needless to say, such a value is not typical of the entire area.

Let us consider bathymetric features which are covered by a minimum of 1000 fathoms of water, so that the first 1000 fathoms below the navigating unit is homogeneous. There are features in the ocean areas which are of such an extent that they may be considered to be of infinite length, and may be approximated as being made of a combination of several bodies which are rectangular in cross section. Heiland (3) has given formulas to compute the attraction of infinite rectangles as follows:



$$\Delta g_z = 2k\sigma\left[H \ln\left[\frac{H^2 + D^2}{H^2 + d^2}\right]^{1/2} - h \ln\left[\frac{D^2 + h^2}{d^2 + h^2}\right]^{1/2} + D\left[\tan^{-1}\frac{H}{D} - \tan^{-1}\frac{h}{D}\right] - d\left[\tan^{-1}\frac{H}{d} - \tan^{-1}\frac{h}{d}\right]\right]$$

where σ is the density difference between the material of the disturbing mass and that of the homogeneous material surrounding it.

The above equation may be rewritten for the horizontal rather than the vertical component of the change in gravity created by the disturbing mass, in that horizontal direction perpendicular to the axis of the disturbing mass, as

$$\Delta g_x = 2k\sigma \left[D \ln \left[\frac{H^2 + D^2}{h^2 + D^2} \right]^{1/2} - d \ln \left[\frac{H^2 + d^2}{h^2 + d^2} \right]^{1/2} + H \left[\tan^{-1} \frac{D}{H} - \tan^{-1} \frac{d}{H} \right] - h \left[\tan^{-1} \frac{D}{h} - \tan^{-1} \frac{d}{h} \right] \right]$$

We will assume that we are working on a plane, so that Δg_x is perpendicular to $\vec{g}$, making it possible to represent the deflection by $\frac{\Delta g_x}{g}$. Three cases have been investigated which are considered as typical of the types of bathymetric features which may be found in mid ocean, and near which operations may presumably be conducted.

CASE I. The Mid-Atlantic Ridge has been approximated by

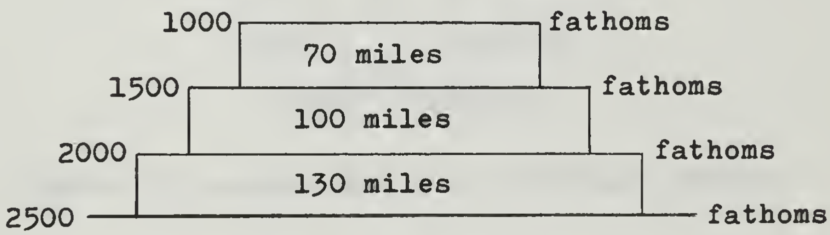


Figure 23. Approximation of Mid-Atlantic Ridge

three blocks of 500 fathom thickness, with widths of 70, 100 and 130 miles, spanning the depths between 1000 and 2500 fathoms, as shown in figure 23.

CASE II. The ridge southeast of the Japanese Islands, near $\varphi = 32^\circ$, $\lambda = 139^\circ$, is similar to the Mid-Atlantic Ridge, but rises out of a more shallow ocean. It has been approximated by two

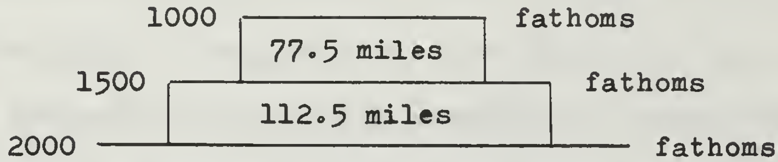


Figure 24. Approximation of ridge south of Japan

blocks of 500 fathom thickness, with widths of 77.5 and 112.5 miles, spanning the depths between 1000 and 2000 fathoms, as shown in figure 24.

CASE III. The northern part of the Marianas Trench, near $\varphi = 32^\circ$, $\lambda = 142^\circ$, drops out of an ocean 3500 fathoms deep to a depth of more than 5000 fathoms. It has been approximated by

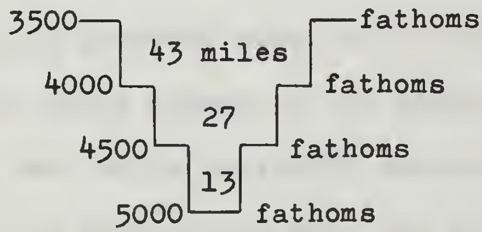


Figure 25. Approximation of Marianas Trench

three blocks, each 500 fathoms thick, with widths of 43, 27 and 13 miles, spanning the depths from 3500 to 5000 fathoms, as shown in figure 25. It is by far the smallest feature of the three considered, and is covered by three and one-half times as thick a layer of homogeneous matter; therefore, we might logically expect

a much smaller deflection in Case III than in Cases I and II.

The magnitude of the deflection in each case is shown in figures 26, 27 and 28, as a function of the distance from the centerline of the disturbing masses. The maximum deflection is, of course, from Case I, being 32.9" at 50 miles from the centerline, with a deflection of over 11" remaining 2° therefrom. In Case II a maximum deflection of 24.7" was found 56 miles from the centerline, with over 7" remaining 2° from the disturbance. Case III, with the minimum effect, had a maximum deflection of 18.7" at 20 miles from the centerline, diminishing to 2.9" at a distance of 2°.

Ordinarily, a deflection value of less that 3" of arc would be considered negligible; however, if we reset the inertial navigator to the true position without considering the deflection of the vertical (i.e., to the geodetic position), after 42 minutes we would have a position geodetic error of 0.1 miles. The importance of this error would depend on the mission of the vessel using the navigator, and, while perfectly acceptable for many missions, this error, in conjunction with the error build-up in practical navigators, could be the cause of the failure of some other mission.

It should be noted that the bathymetry used in Case II and Case III occur with a longitude difference of only 3°, or a spherical separation of about 2.5°. If the navigator were between the two disturbances, the effects would be additive in magnitude. In each of these cases we have considered the effect of only one

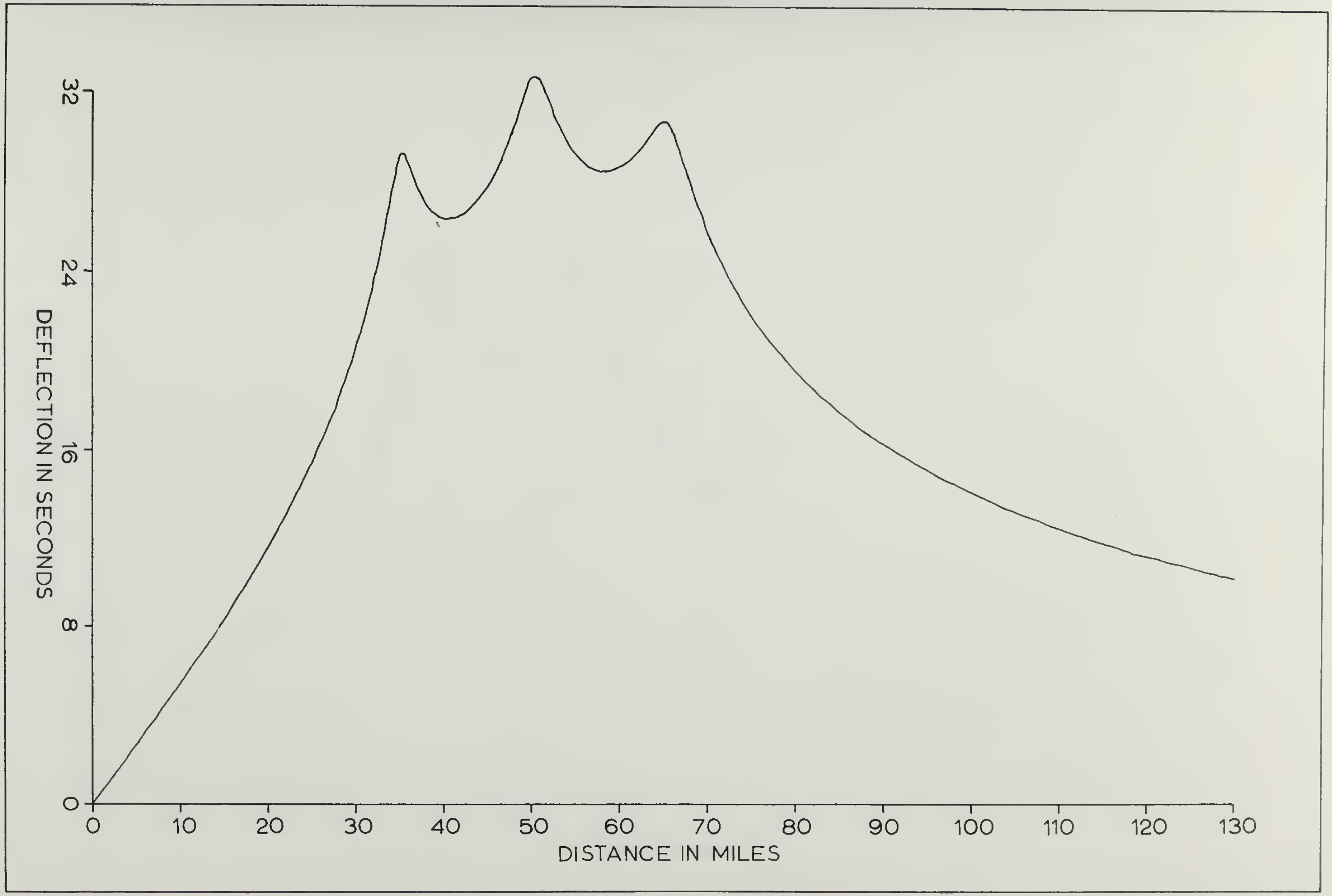


Figure 26. Magnitude of deflections caused by topography of figure 23

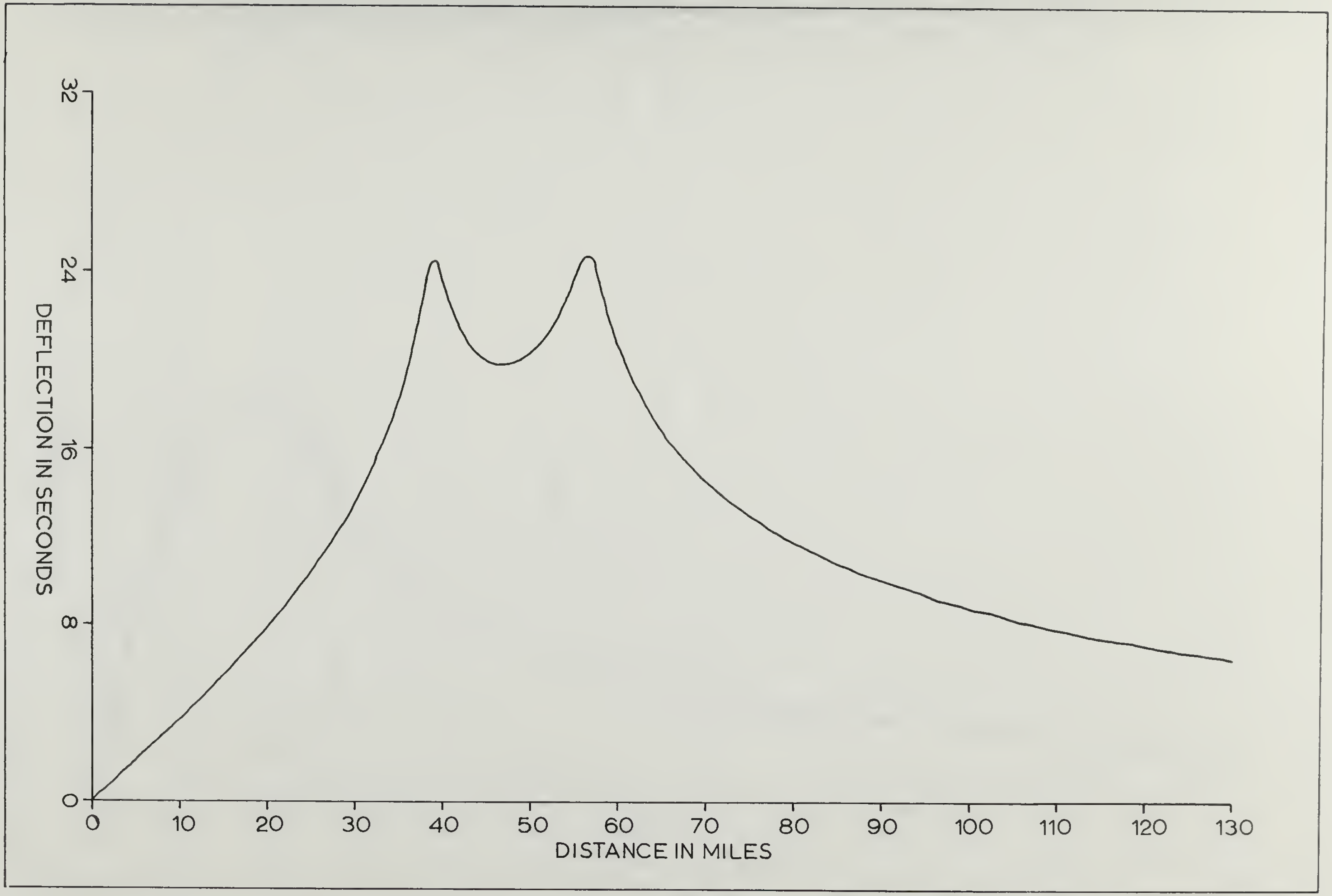


Figure 27. Magnitude of deflections caused by topography of figure 24

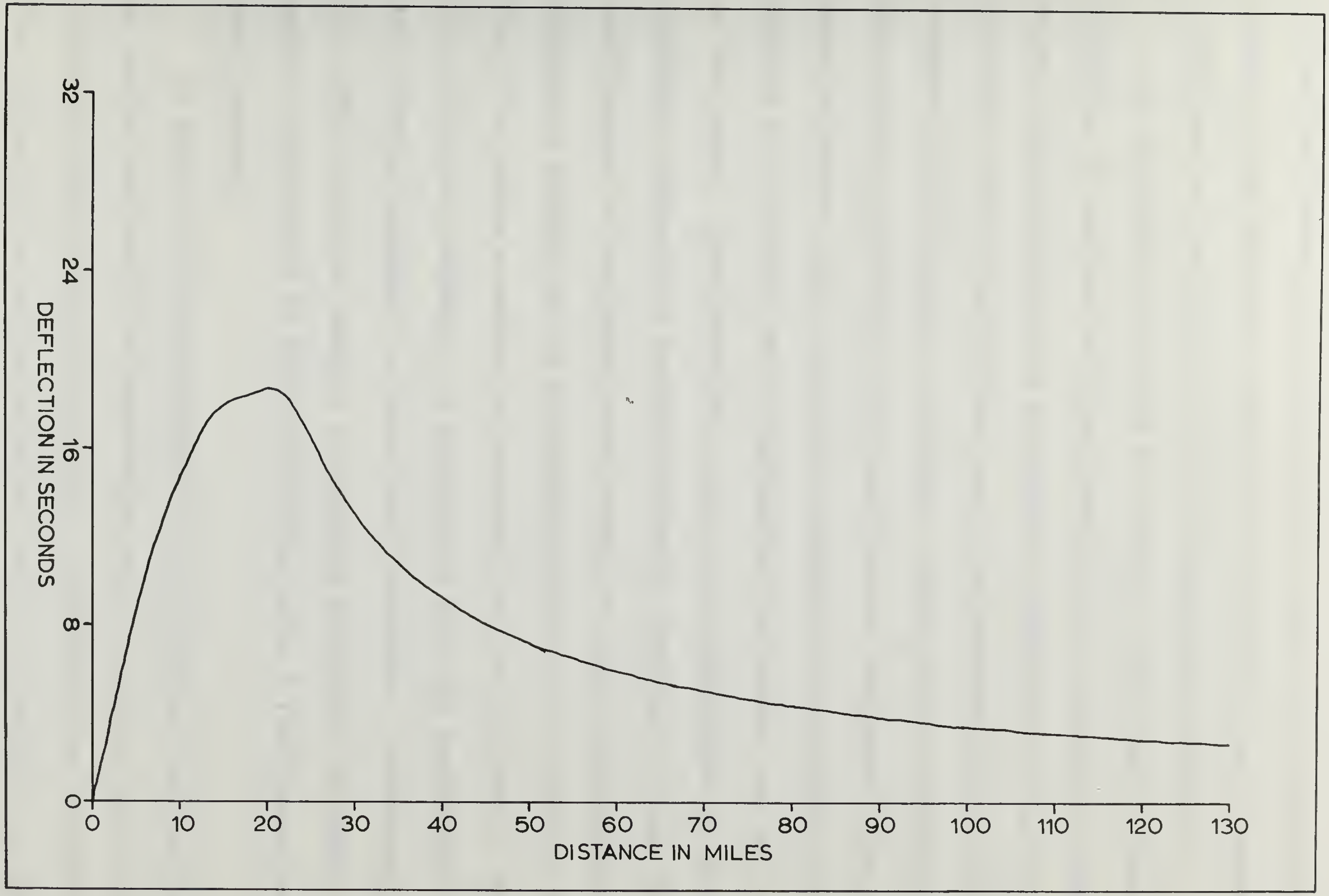


Figure 28. Magnitude of deflections caused by topography of figure 25

disturbing feature, and have completely ignored the compensating effect of any isostatic compensation, which would be of increasing importance as the distance from the disturbance increased.

If we consider the world as a whole, rather than isolated features, we may work with the gravitational potential as determined from satellites, and see what the undulations of an equipotential surface derived from satellite observations are with respect to an ellipsoid of rotation. First, let us recognize that the potential determined from satellites at altitudes of one or more megameters cannot adequately describe the potential on the surface of the earth, so the undulations determined are not those of the geoid. They represent, rather, the undulations of an equipotential surface of a harmonic potential function in which the coefficients of the higher order terms have been inadequately determined because of the attenuating effect of the distance between the masses creating the potential and the altitude at which the effects are measured, in conjunction with the inadequacy of the tracking equipment to detect small movements. Additionally, the insufficiency of numbers of tracking equipments to cover the entire trajectories of these satellites gives a fictitious potential function.

In spite of the shortcomings of the potential function which we obtain from satellite observations when we want to apply that function at the surface of the earth, the undulations obtained from the satellite derived potential should correspond in a gross way to those obtained from the gravimetric method, and the slope

of the undulating surface with respect to an approximating ellipsoid should agree with the average deflections of the vertical obtained by the gravimetric method when compared in an area of great enough extent, as determined by the degree to which the satellite potential is accurately computed. Thus, if we were to compute the slope of the satellite potential surface relative to the ellipsoid, we would not expect to agree in any measure with the deflection of the vertical as obtained from gravimetry; however, if we obtain the average slope of this surface over an extended distance, we would expect fair agreement with the average of the deflections of the vertical over the same distance.

The gravitational field representation developed by Anderle (1), which is developed through the seventh degree, sixth order, was converted to undulations as outlined by Nicolaides and Macomber (7). The formula for the undulations is

$$\zeta = \rho\left[\sum_{n=2}^{7} \sum_{m=0}^{n} P_n^m(\sin\bar{\varphi})\left[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right] - J_2 P_2(\sin\bar{\varphi}) - J_4 P_4(\sin\bar{\varphi})\right]$$

where

ρ is the radius vector

P_n are Legendre Polynomials

P_n^m are Associate Legendre Functions

C_{nm} and S_{nm} are coefficients in the gravitational potential expansion, and

J_2 and J_4 apply to the reference ellipsoid

The J_2 and J_4 used were for a reference ellipsoid flattening of 1/298.3. From these undulations, deflections were computed from the equations

$$\xi = \frac{\Delta\zeta}{M\ \Delta\varphi}$$

$$\eta = \frac{\Delta\zeta}{N\ \cos\varphi\ \Delta\lambda}$$

The results of this analysis showed that in the North Atlantic Ocean, in a region bounded on the south by $\varphi = 30^\circ$, between the meridians of -30° and -40°, and on the north by $\varphi = 70^\circ$, between the meridians of -30° and -80°, the average slope is on the order of seven and one-half seconds of arc. This is, admittedly, the largest ocean area which shows a systematic slope of this order of magnitude. If a perfect inertial navigator were reset to the geodetic position at a spot where the average deflection were applicable, the position geodetic error after a period of 42 minutes would be one-quarter mile. Since the average error is of the order of seven and one-half seconds, it is reasonable to expect some areas where the deflection is larger, resulting in position geodetic errors larger than one-fourth mile if the navigator were reset without compensating for the effect of the deflection of the local vertical.

There is nothing in this chapter which should be interpreted to indicate that deflections of the vertical of any given magnitude do exist in the ocean areas. The purpose of this chapter is

to show that one cannot reasonably exclude the possibility that deflections exist in deep ocean areas, which are of such a magnitude that they could, if ignored, introduce sufficient error into an inertial navigator to cause the failure of some mission which requires precise navigation for its successful accomplishment.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

A mechanically perfect inertial navigator is extremely sensitive to the effects of the magnitude of the gravity vector if an attempt is made to navigate along that vector. A slight error in the magnitude of gravity will cause an unbounded error to develop in the vertical direction. This unbounded error source makes the inertial navigator totally unsuited for the navigation task in this direction. If we consider a case in which there is an independent input for height information, and the inertial navigator is used only for navigation in the horizontal plane, the magnitude of gravity is of no practical importance.

When we use a mathematical model for our navigation problem which is based on an ellipsoid of rotation for the figure of the earth, with position defined by the normal to that ellipsoid, we find that the inertial navigator seeks the local vertical rather than the normal to the fictitious reference ellipsoid. If we set the navigator to the geodetic position, the navigator's position will oscillate through twice the angle of the deflection of the vertical, with a period of a Schuler pendulum, and the navigator's position will lie near a plane rotating about the astronomic position with the angular velocity of a Foucault pendulum relative to the earth.

If we had a perfect inertial navigator, and set the position initially to the astronomic position while stationary with respect to the earth, then the navigator would provide us continually with positions within a few meters of the astronomic positions as we traversed an area with varying deflections of the vertical.

Unfortunately, a mechanically perfect inertial navigator has not been developed, so that errors do creep into the system, making it necessary to reset the navigator periodically. Resetting to the geodetic position is unsatisfactory since the navigator's position will then oscillate with twice the amplitude of the deflection of the vertical. In order to avoid this oscillation and at the same time retain the current level of sophistication in the computer programs, it is mandatory that the position for resetting is that defined as the astronomic position, and that the angular velocities we accept at the time of resetting be the astronomic latitude rate and the astronomic longitude rate.

If accurate navigation is required through the use of an inertial navigator, it is necessary that some means be available for resetting position to that of the astronomic position and angular velocity to the astronomic rates, or, alternatively, to know the gravity vector precisely, and to compensate for the geodetically horizontal components if the navigator is reset to the geodetic position and the angular velocity is reset to the geodetic latitude and longitude rates. The alternative would necessitate a great increase in the computation load for determining positions.

The ideal case for resetting the navigator, from the standpoint of simplicity, is to do so in an area where both the deflection of the vertical and the time rate of change of the local vertical are zero. It is realized that this condition may be hard to achieve operationally.

It is recommended that, where a valid requirement exists for precise navigation, and where an inertial navigator is to satisfy that requirement, that a complete gravimetric survey of the area be made, and that charts showing the deflection of the vertical be prepared, with entering arguments of geodetic position in one case, and of astronomic position in the other. This will permit taking a geodetic position and converting it to astronomic when resetting the navigator, and to compute the time rate of change of the components of the deflection of the vertical which are necessary in determining the astronomic latitude and longitude rates, and then taking an updated navigator's position and converting it to an approximate geodetic position when needed.

The fact that the effects of non-normal gravity were considered on a mechanically perfect inertial navigator should not be construed to indicate that the writer feels that the mechanical imperfections are unimportant and can be ignored. Rather, he feels incompetent to consider them without extensive collaboration with engineers specializing in the field. Now that the effects of non-normal gravity have been investigated while being isolated, it would be most appropriate to undertake a study combining the effects of non-normal gravity and mechanical imper-

fections, so that it may be seen where the influence of one overshadows the influence of the other.

Although the underlying assumption throughout this study has been that information on the deflection of the vertical would be obtained through gravimetric studies, one should not rule out the possibility of developing a sensor which would be able to distinguish between the mass attraction effects, the centrifugal force, and the reaction to acceleration. Such a sensor would change these conclusions and recommendations completely. Similarly, if a sensor could be developed which would continuously analyze the gradients of gravity to carry either absolute or relative deflections with the navigating vessel, the recommendations would have to be modified accordingly.

APPENDIX I

DEVELOPMENT OF FORMULAS FOR SENSORS FIXED IN INERTIAL SPACE

The Y coordinate system will be used in this case. The accelerometers, which sense along the coordinate axes, respond to mass attraction and accelerations. Streeter (10) has shown that accelerations due to rotation about the sun can be neglected and that the mass attractions of the matter outside the earth are negligible. Letting $\vec{A}_s$ represent the value sensed by the accelerometers, we may write

$$\vec{A}_s = \vec{A}_{NI} - \vec{G} \qquad \text{I.1}$$

$$= \vec{A}_{NB} + \vec{A}_{BI} - \vec{G}$$

where $\vec{A}_{NI}$ = the acceleration of the navigator with respect to inertial space,

$\vec{A}_{NB}$ = the acceleration of the navigator with respect to point B, said point being fixed to the earth on the surface of the reference ellipsoid, and

$\vec{A}_{BI}$ = the acceleration of point B with respect to inertial space.

Since point B is fixed to the reference ellipsoid, the acceleration is merely centripetal:

$$\vec{A}_{BI} = \vec{\Omega}\times(\vec{\Omega}\times\vec{R}) = \vec{\Omega}\times(\vec{\Omega}\times\vec{N}_B)$$

where $\vec{N}_B$ is the vector from the spin axis to point B which is normal to the reference ellipsoid at point B.

$$\vec{A}_S = \vec{\Omega}\times(\vec{\Omega}\times\vec{N}_B) + \vec{A}_{NB} - \vec{G} \qquad \text{I.2}$$

In the problem of practical navigation, the only quantity measured is the sensed value provided by the accelerometer. Since the value for the gravitational vector is imperfectly known, ~~it~~ is approximated by "normal" gravitation, providing, in general, a false indication of the acceleration of the navigator, $\ddot{\vec{R}}_{NB}$.

$$\vec{A}_S = \vec{\Omega}\times(\vec{\Omega}\times\vec{N}_B) + \ddot{\vec{R}}_{NB} - \vec{G}_N$$

$$\ddot{\vec{R}}_{NB} = \vec{A}_{NB} - \vec{G} + \vec{G}_N \qquad \text{I.3}$$

Normal gravitation has been developed (Hirvonen and Moritz, (5)) in terms of the geocentric radius, R, the geocentric latitude, $\bar{\varphi}$, and the longitude, λ, once the defining parameters of the gravitational reference ellipsoid are known. For the International Ellipsoid with the International normal formula of gravity, when R is expressed in kilometers and G in gals, we obtain the following:

$$\begin{aligned} G_R = -\,1000\,F &+ 0.835\,888\,F^2 - 2.507\,664\,F^2\cos 2\bar{\varphi} + \\ &+ 0.005\,118\,F^3 + 0.003\,937\,F^3\cos 2\bar{\varphi} - \\ &- 0.013\,778\,F^3\cos^2 2\bar{\varphi} - \\ &- 0.000\,007\,F^4 + 0.000\,054\,F^4\cos 2\bar{\varphi} + \\ &+ 0.000\,023\,F^4\cos^2 2\bar{\varphi} - 0.000\,085\,F^4\cos^3 2\bar{\varphi}, \end{aligned}$$

where $F = \frac{3986\ 3290.45}{R^2}$ and G_R is the component of normal gravi-

tation along the geocentric radius vector. The component in the meridional plane orthogonal to the radius vector, $G_{\bar{\varphi}}$, is

$$G_{\bar{\varphi}} = \sin\bar{\varphi}\cos\bar{\varphi}\Big[-3.343\,551\,F^2 + 0.003\,149\,F^3 - 0.022\,045\,F^3\cos2\bar{\varphi} + 0.000\,031\,F^4 + 0.000\,026\,F^4\cos2\bar{\varphi} - 0.000\,145\,F^4\cos^2 2\bar{\varphi}\Big]$$

The components of gravitation along the earth-fixed coordinate axes may then be obtained from

$$G_{X^0} = \left[G_R\cos\bar{\varphi} - G_{\bar{\varphi}}\sin\bar{\varphi}\right]\cos\lambda$$

$$G_{X^1} = \left[G_R\cos\bar{\varphi} - G_{\bar{\varphi}}\sin\bar{\varphi}\right]\sin\lambda.$$

$$G_{X^2} = G_R\sin\bar{\varphi} + G_{\bar{\varphi}}\cos\bar{\varphi}$$

The value of the gravitation vector actually existing at a given point is the sum of the normal gravitation vector and the gravity disturbance vector, $\vec{\delta}$, at that point. The gravity disturbance vector is computed as shown in Hirvonen and Moritz (5), and transformed to the coordinate axes in the same way as the normal gravitation vector.

After we have added the components of the normal gravitation, $\vec{G}$, and the components of the gravity disturbance vector, $\vec{\delta}$, it is necessary to transform them from the earth-fixed coordinate system, X, to the inertial coordinate system, Y, by considering the angle through which the earth has turned, Ωt. We then put the transformed components into equation I.3 to obtain the value which the

inertial navigator interprets to be his acceleration in inertial space, $\ddot{\vec{R}}_{NB}$.

$$\left.\begin{aligned}\ddot{\vec{R}}_{NB} = &\left[\ddot{Y}^{0} + (G_{X^0} + \delta_{X^0} - G_{X^0_N})\cos\Omega t - (G_{X^1} + \delta_{X^1} - G_{X^1_N})\sin\Omega t\right]\vec{i} \\ &+\left[\ddot{Y}^{1} + (G_{X^0} + \delta_{X^0} - G_{X^0_N})\sin\Omega t + (G_{X^1} + \delta_{X^1} - G_{X^1_N})\cos\Omega t\right]\vec{j} + \\ &+\left[\ddot{Y}^{2} + (G_{X^2} + \delta_{X^2} - G_{X^2_N})\right]\vec{k}\end{aligned}\right\} \text{I.4}$$

In the present case, wherein the vehicle to be navigated is a surface vessel and remains close to a known equipotential surface rather than navigating freely in three dimensions, we may simplify the procedure for computing both the normal and the actual gravitation by using gravity anomalies, Δg, rather than gravity disturbances, δg. The magnitude of the actual gravity on the geopotential surface $W = W_0$ is normal gravity, computed on the spheropotential surface $U = W_0$, plus the gravity anomaly. The direction in which it acts is defined by the angles $\varphi + \xi$ and $\lambda + \eta/\cos\varphi$.

$$\vec{G} = \vec{g} + \vec{\Omega}\times(\vec{\Omega}\times\vec{R})$$

Then

$$\begin{aligned}\vec{G}_N = &-(\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N \cos(\lambda_N + \Omega t)\ \vec{i} \\ &-(\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N \sin(\lambda_N + \Omega t)\ \vec{j} \\ &- \gamma_N \sin\varphi_N\ \vec{k}\end{aligned}$$

Similarly, we may write the expression for the actual gravitation and account for the non-normality of field.

$$\vec{G} = -\Big[(\gamma + \Delta g)\cos(\varphi + \xi)\ \cos(\lambda + \eta/\cos\varphi + \Omega t) +$$

$$\Omega^2 (N + h)\cos\varphi\ \cos(\lambda + \Omega t)\Big]\ \vec{i}$$

$$-\Big[(\gamma + \Delta g)\cos(\varphi + \xi)\ \sin(\lambda + \eta/\cos\varphi + \Omega t) +$$

$$\Omega^2 (N + h)\cos\varphi\ \sin(\lambda + \Omega t)\Big]\ \vec{j}$$

$$- (\gamma + \Delta g)\sin(\varphi + \xi)\ \ \vec{k}$$

Substituting these expressions for the actual and the normal gravitation into equation I.3, we obtain an expression for the value of the acceleration used in the navigation problem.

$$\left.\begin{aligned}
\ddot{\vec{R}}_{NB} = &\Big[\ddot{Y}^0 + (\gamma + \Delta g)\cos(\varphi + \xi)\ \cos(\lambda + \eta/\cos\varphi + \Omega t) + \\
&+ \Omega^2 (N + h)\cos\varphi\ \cos(\lambda + \Omega t) - \\
&- (\gamma_N + \Omega^2 (N_N + h_N))\cos\varphi_N\ \cos(\lambda_N + \Omega t)\Big]\ \vec{i} + \\
&+\Big[\ddot{Y}^1 + (\gamma + \Delta g)\cos(\varphi + \xi)\ \sin(\lambda + \eta/\cos\varphi + \Omega t) + \\
&+ \Omega^2 (N + h)\cos\varphi\ \sin(\lambda + \Omega t) - \\
&- (\gamma_N + \Omega^2 (N_N + h_N))\cos\varphi_N\ \sin(\lambda_N + \Omega t)\Big]\ \vec{j} + \\
&+\Big[\ddot{Y}^2 + (\gamma + \Delta g)\sin(\varphi + \xi) - \gamma_N \sin\varphi_N\Big]\ \vec{k}
\end{aligned}\right\}\ \text{I.5}$$

The gravity formula used in this study is the formula for the International Ellipsoid, modified to include a term accounting for free-air reduction where the height is positive, and a term accounting for the free-air reduction plus the effect of salt

water in those cases where the height is negative. The formulas used are:

a. For h positive -

$$\gamma = 978.049\ (1 + 0.0052884 \sin^2 \varphi - 0.0000059 \sin^2 2\varphi) - $$
$$- 0.3086\ h \times 10^{-3} \text{ centimeters/sec}^2$$

b. For h negative -

$$\gamma = 978.049\ (1 + 0.0052884 \sin^2 \varphi - 0.0000059 \sin^2 2\varphi) - $$
$$- 0.2223\ h \times 10^{-3} \text{ centimeters/sec}^2$$

The determination of position in such a rectangular system is very straight-forward, and may be developed in the following manner.

$$\vec{R} = \vec{R}_0 + \mathring{\vec{R}}_0 t + \int_0^t \int_0^v \ddot{\vec{R}}\, du\, dv$$

$$\vec{R} = \vec{R}_0 + \mathring{\vec{R}}_{0\,NB}\, t + \mathring{\vec{R}}_{0\,BI}\, t +$$
$$+ \int_0^t \int_0^v \ddot{\vec{R}}_{NB}\, du\, dv + \int_0^t \int_0^v \ddot{\vec{R}}_{BI}\, du\, dv$$

Introducing the components of the initial value of the radius vector, the initial velocity of the navigator with respect to the ellipsoid and the values from equation I.5, we obtain an expression for the radius vector at any time as shown below.

$$\vec{R} = \left.\begin{array}{l} \begin{bmatrix} (N_0 + h_0)\cos\varphi_0 \cos\lambda_0 + \dot{Y}^0_{0\,NB}\, t + \int_0^t \int_0^v \ddot{Y}^0_{NB}\, du\, dv \\ (N_0 + h_0)\cos\varphi_0 \sin\lambda_0 + \dot{Y}^1_{0\,NB}\, t + \int_0^t \int_0^v \ddot{Y}^1_{NB}\, du\, dv \\ (N_0(1-e^2) + h_0)\sin\varphi_0 + \dot{Y}^2_{0\,NB}\, t + \int_0^t \int_0^v \ddot{Y}^2_{NB}\, du\, dv \end{bmatrix} \\ + (\vec{\Omega}\times\vec{N}_B)t + \int_0^t \int_0^v \vec{\Omega}\times(\vec{\Omega}\times\vec{N}_B)\, du\, dv \end{array}\right\} \quad \text{I.6}$$

The last two terms on the right-hand side of the equation carry point B in a circle about the earth's spin axis, and can be accounted for by a simple rotation about the Y^2 axis. For an incremental time step of Δt we may write

$$R_2(-\Omega\,\Delta t) = \begin{bmatrix} \cos(\Omega\,\Delta t) & -\sin(\Omega\,\Delta t) & 0 \\ \sin(\Omega\,\Delta t) & \cos(\Omega\,\Delta t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Dropping the subscript $_{NB}$ as being superfluous once the motion of point B with respect to inertial space is accounted for, and introducing the symbol $\Delta\vec{R} = \vec{R}_N - \vec{R}_B$, we can rewrite equation I.6 in terms of incremental time steps, relating the position at time $t + \Delta t$ to the position at time t as the equations which follow indicate.

$$\vec{R}_{(j+1)} = R_3(-\Omega\,\Delta t)\,\vec{R}_{s\,(j)} + \Delta\vec{R}_{(j)} + \dot{\vec{R}}_{(j)}\Delta t + $$

$$+ \frac{1}{4}(\Delta t)^2 \left(\ddot{\vec{R}}_{(j+1)} + \ddot{\vec{R}}_{(j)}\right)$$

or

$$\left.\begin{aligned}
\begin{bmatrix} Y^0 \\ Y^1 \\ Y^2 \end{bmatrix}_{(j+1)} &= R_3(-\Omega\,\Delta t) \begin{bmatrix} Y_s^0 \\ Y_s^1 \\ Y_s^2 \end{bmatrix}_{(j)} + \begin{bmatrix} \Delta Y^0 \\ \Delta Y^1 \\ \Delta Y^2 \end{bmatrix}_{(j)} + \begin{bmatrix} \dot{Y}^0 \\ \dot{Y}^1 \\ \dot{Y}^2 \end{bmatrix}_{(j)} \Delta t + \\
&+ \frac{1}{4}(\Delta t)^2 \begin{bmatrix} \ddot{Y}^0{}_{(j+1)} + \ddot{Y}^0{}_{(j)} \\ \ddot{Y}^1{}_{(j+1)} + \ddot{Y}^1{}_{(j)} \\ \ddot{Y}^2{}_{(j+1)} + \ddot{Y}^2{}_{(j)} \end{bmatrix}
\end{aligned}\right\} \quad \text{I.7}$$

Since we cannot, in practice, measure $\vec{A}$ ($\ddot{Y}^0$, $\ddot{Y}^1$, $\ddot{Y}^2$), we must substitue components of $\ddot{\vec{R}}$ which we derive from $\vec{A}_s$ and which contain the effects of non-normal gravitation. Thus we use quantities obtained from the values sensed by the accelerometers. In this study we shall construct the sensed values by taking true acceleration, adding the true (hypothetical) gravitation, and then subtracting the normal gravitation. Substituting the quantities so derived into equations I.7 we obtain the set of equations shown below.

$$\left.\begin{aligned}
\begin{bmatrix} Y^0 \\ Y^1 \\ Y^2 \end{bmatrix}_{(j+1)} &= R_2(-\Omega\,\Delta t)\begin{bmatrix} Y_s^0 \\ Y_s^1 \\ Y_s^2 \end{bmatrix}_{(j)} + \begin{bmatrix} \Delta Y^0 \\ \Delta Y^1 \\ \Delta Y^2 \end{bmatrix}_{(j)} + \begin{bmatrix} \mathring{Y}^0 \\ \mathring{Y}^1 \\ \mathring{Y}^2 \end{bmatrix}_{(j)} \Delta t \, + \\
&+ \tfrac{1}{4}(\Delta t)^2 \begin{bmatrix} \ddot{Y}^0 + (\gamma + \Delta g)\cos(\varphi + \xi)\cos(\lambda + \eta/\cos\varphi + \Omega t) + \Omega^2(N + h)\cos\varphi\cos(\lambda + \Omega t) \\ \ddot{Y}^1 + (\gamma + \Delta g)\cos(\varphi + \xi)\sin(\lambda + \eta/\cos\varphi + \Omega t) + \Omega^2(N + h)\cos\varphi\sin(\lambda + \Omega t) \\ \ddot{Y}^2 + (\gamma + \Delta g)\sin(\varphi + \xi) \quad \ldots \end{bmatrix}_{(j)} + \\
&+ \tfrac{1}{4}(\Delta t)^2 \begin{bmatrix} \ddot{Y}^0 + (\gamma + \Delta g)\cos(\varphi + \xi)\cos(\lambda + \eta/\cos\varphi + \Omega t) + \Omega^2(N + h)\cos\varphi\cos(\lambda + \Omega t) \\ \ddot{Y}^1 + (\gamma + \Delta g)\cos(\varphi + \xi)\sin(\lambda + \eta/\cos\varphi + \Omega t) + \Omega^2(N + h)\cos\varphi\sin(\lambda + \Omega t) \\ \ddot{Y}^2 + (\gamma + \Delta g)\sin(\varphi + \xi) \quad \ldots \end{bmatrix}_{(j+1)} -
\end{aligned}\right\} \quad \text{I.8}$$

$$- \frac{1}{4}(\Delta t)^2 \begin{bmatrix} (\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N \cos(\lambda_N + \Omega t) \\ (\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N \sin(\lambda_N + \Omega t) \\ \gamma_N \sin\varphi_N \end{bmatrix}_{(j)} -$$

$$- \frac{1}{4}(\Delta t)^2 \begin{bmatrix} (\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N \cos(\lambda_N + \Omega t) \\ (\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N \sin(\lambda_N + \Omega t) \\ \gamma_N \sin\varphi_N \end{bmatrix}_{(j+1)} \Bigg]$$

where Δt is sufficiently small that linear interpolation between the components of gravitation at $t_{(j)}$ and $t_{(j+1)}$ is permissible. In actual navigation, where a computer is allocated to the navigation task and position is updated several times a second, an approach similar to the above is acceptable; however, rather than use a small time increment and compute many points per second which are not needed in this study, the effect of curvature in the quantities to be integrated will be considered.

For the purposes of this study, the true initial velocity and the true acceleration of the navigator with respect to the earth will be zero. The latitude and geographic longitude rates of the navigator's position will be the result of variations in the gravitation vector, and should be small. Equation I.8 can then be written, for use with a larger incremental time step in the form shown below.

$$
\left.\begin{aligned}
&\begin{bmatrix} Y^0 \\ Y^1 \\ Y^2 \end{bmatrix}_{(j+1)} = R_2(-\Omega\,\Delta t)\begin{bmatrix} Y^0_B \\ Y^1_B \\ Y^2_B \end{bmatrix}_{(j)} + \begin{bmatrix} \Delta Y^0 \\ \Delta Y^1 \\ \Delta Y^2 \end{bmatrix}_{(j)} + \begin{bmatrix} \dot{Y}^0 \\ \dot{Y}^1 \\ \dot{Y}^2 \end{bmatrix}_{(j)} \Delta t \; + \\
&+ \frac{1}{2}(\gamma + \Delta g)\begin{bmatrix} \cos(\varphi + \xi)\ \text{Av.Val.}\left[\cos(\lambda + \eta/\cos\varphi + \Omega t)\right] \\ \cos(\varphi + \xi)\ \text{Av.Val.}\left[\sin(\lambda + \eta/\cos\varphi + \Omega t)\right] \\ \sin(\varphi + \xi) \end{bmatrix}(\Delta t)^2 \; + \\
&+ \frac{1}{2}\Omega^2 (N + h)\cos\varphi \begin{bmatrix} \text{Av.Val.}\left[\cos(\lambda + \Omega t)\right] \\ \text{Av.Val.}\left[\sin(\lambda + \Omega t)\right] \\ 0 \end{bmatrix}(\Delta t)^2 \; - \\
&- \frac{1}{2}\begin{bmatrix} \text{Av.Val.}\left[(\gamma_N + \Omega^2 (N_N + h_N))\cos\varphi_N \ \cos(\lambda_N + \Omega t)\right] \\ \text{Av.Val.}\left[(\gamma_N + \Omega^2 (N_N + h_N))\cos\varphi_N \ \sin(\lambda_N + \Omega t)\right] \\ \text{Av.Val.}\left[\gamma_N \sin\varphi_N\right] \end{bmatrix}(\Delta t)^2
\end{aligned}\right\} \quad \text{I.9}
$$

The most complex of the average value expressions, $(\gamma_N + \Omega^2 (N_N + h_N))\cos\varphi_N \begin{bmatrix}\cos \\ \sin\end{bmatrix} (\lambda_N + \Omega t)$ can be written as $(f_1(\varphi,h) + f_2(\varphi,h))f_3(\varphi)f_4(\lambda + \Omega t) = F_1(\varphi,h)f_4(\lambda + \Omega t)$. Since φ and h are orthogonal to $\lambda + \Omega t$, we know that the average value of the product $F_1(\varphi,h)f_4(\lambda + \Omega t)$ is the product of the average values. The quantities φ_N and h_N will change, at least initially, very slowly because of the initial conditions imposed. We may take

advantage of this fact and write the average value of $F_1(\varphi, h)$ as

$$\text{Av.Val.}\left[(\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N\right] =$$

$$= \frac{1}{2}\left[(\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N\right]_{(j)}$$

$$+ \frac{1}{2}\left[(\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N\right]_{(j+1)}, \text{ and}$$

$$\text{Av.Val.}\left[\gamma_N \sin\varphi_N\right] = \frac{1}{2}(\gamma_N \sin\varphi_N)_{(j)} + \frac{1}{2}(\gamma_N \sin\varphi_N)_{(j+1)}$$

For the average value of the longitude term we use the fact that the average value of some function, f(x), over an interval from a to b equals

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$\frac{1}{\Delta\lambda + \Omega\Delta t}\int_{\lambda + \Omega t}^{\lambda + \Omega t + \Delta\lambda + \Omega\Delta t} \sin x\,dx =$$

$$= \frac{1}{\Delta\lambda + \Omega\Delta t}\left[-\cos(\lambda + \Omega t + \Delta\lambda + \Omega\Delta t) + \cos(\lambda + \Omega t)\right] =$$

$$= \frac{1}{\Delta\lambda + \Omega\Delta t}\left[\cos(\lambda + \Omega t) - \cos(\lambda + \Omega t)\cos(\Delta\lambda + \Omega\Delta t) + \right.$$

$$\left. + \sin(\lambda + \Omega t)\sin(\Delta\lambda + \Omega\Delta t)\right] =$$

$$= \frac{1}{\Delta\lambda + \Omega\Delta t}\Big[\cos(\lambda + \Omega t)(1 - \cos(\Delta\lambda + \Omega\Delta t)) +$$

$$+ \sin(\lambda + \Omega t)\sin(\Delta\lambda + \Omega\Delta t)\Big] =$$

$$= \cos(\lambda + \Omega t)\left[\frac{\Delta\lambda + \Omega\Delta t}{2!} - \frac{(\Delta\lambda + \Omega\Delta t)^3}{4!} + \ldots\right] +$$

$$+ \sin(\lambda + \Omega t)\left[1 - \frac{(\Delta\lambda + \Omega\Delta t)^2}{3!} + \frac{(\Delta\lambda + \Omega\Delta t)^4}{5!} - \ldots\right]$$

Because of the initial conditions imposed, $\Delta\lambda \approx 0$, so $\Delta\lambda + \Omega\Delta t \approx \Omega\Delta t \approx 0.005$ for $\Delta t = 60$ seconds.

$$(\Omega\Delta t)^3 \approx 10^{-7} \qquad (\Omega\Delta t)^4 \approx 10^{-9}$$

$$\frac{(\Omega\Delta t)^3}{4!} \approx 10^{-8} \qquad \frac{(\Omega\Delta t)^4}{5!} \approx 10^{-11}$$

Since the error of truncation of a series of this type is less than the first neglected term, we are safe in going only to the cube of $\Delta\lambda + \Omega\Delta t$.

$$\text{Av.Val.}\Big[\sin(\lambda + \Omega t)\Big] = \sin(\lambda + \Omega t)_{(j)}\left[1 - \frac{(\Delta\lambda + \Omega\Delta t)^2}{6}\right] +$$

$$+ \cos(\lambda + \Omega t)_{(j)}\,\frac{\Delta\lambda + \Omega\Delta t}{2}\left[1 - \frac{(\Delta\lambda + \Omega\Delta t)^2}{12}\right]$$

Similarly,

$$\text{Av.Val.}\Big[\cos(\lambda + \Omega t)\Big] = \cos(\lambda + \Omega t)_{(j)}\left[1 - \frac{(\Delta\lambda + \Omega\Delta t)^2}{6}\right] -$$

$$- \sin(\lambda + \Omega t)_{(j)}\,\frac{\Delta\lambda + \Omega\Delta t}{2}\left[1 - \frac{(\Delta\lambda + \Omega\Delta t)^2}{12}\right]$$

Substituting the average values of the longitude and latitude functions into equation I.9 we obtain the final equations for rectangular coordinates of position in an inertial reference frame as shown below.

$$\left.\begin{aligned}
&\begin{bmatrix} Y^0 \\ Y^1 \\ Y^2 \end{bmatrix}_{(j+1)} = \begin{bmatrix} \cos\Omega\Delta t & -\sin\Omega\Delta t & 0 \\ \sin\Omega\Delta t & \cos\Omega\Delta t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y_s^0 \\ Y_s^1 \\ Y_s^2 \end{bmatrix}_{(j)} + \begin{bmatrix} \Delta Y^0 \\ \Delta Y^1 \\ \Delta Y^2 \end{bmatrix}_{(j)} + \\
&+ \begin{bmatrix} \dot{Y}^0 \\ \dot{Y}^1 \\ \dot{Y}^2 \end{bmatrix}_{(j)} \Delta t + \begin{bmatrix} \cos(\varphi + \xi)\left(\cos(\lambda + \frac{\eta}{\cos\varphi} + \Omega t)\left[1 - \frac{(\Omega\Delta t)^2}{6}\right] - \right. \\ \cos(\varphi + \xi)\left(\sin(\lambda + \frac{\eta}{\cos\varphi} + \Omega t)\left[1 - \frac{(\Omega\Delta t)^2}{6}\right] + \right. \\ \sin(\varphi + \xi) \end{bmatrix} \\
&\begin{bmatrix} \left. - \sin(\lambda + \frac{\eta}{\cos\varphi} + \Omega t)\frac{\Omega\Delta t}{2}\left[1 - \frac{(\Omega\Delta t)^2}{12}\right]\right) \\ \left. + \cos(\lambda + \frac{\eta}{\cos\varphi} + \Omega t)\frac{\Omega\Delta t}{2}\left[1 - \frac{(\Omega\Delta t)^2}{12}\right]\right) \\ \cdots \end{bmatrix} \frac{1}{2}(\gamma + \Delta g)(\Delta t)^2 + \\
&+ \frac{1}{2}\Omega^2(N + h)\cos\varphi\,(\Delta t)^2 \begin{bmatrix} \cos(\lambda + \Omega t)\left[1 - \frac{(\Omega\Delta t)^2}{6}\right] - \\ \sin(\lambda + \Omega t)\left[1 - \frac{(\Omega\Delta t)^2}{6}\right] + \\ 0 \end{bmatrix}
\end{aligned}\right\} \quad \text{I.10}$$

$$- \sin(\lambda + \Omega t)\ \frac{\Omega\Delta t}{2}\left[1 - \frac{(\Omega\Delta t)^2}{12}\right]$$

$$+ \cos(\lambda + \Omega t)\ \frac{\Omega\Delta t}{2}\left[1 - \frac{(\Omega\Delta t)^2}{12}\right]\Bigg] -$$

. . .

$$- \frac{1}{4}(\Delta t)^2 \Bigg[\Big(\Big[(\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N\Big]_{(j)} +$$

$$\Big(\Big[(\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N\Big]_{(j)} +$$

$$\Big[\gamma_N \sin\varphi_N\Big]_{(j)} + \Big[\gamma_N \sin\varphi_N\Big]_{(j+1)}$$

$$+ \Big[(\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N\Big]_{(j+1)}\Big)\ \Big(\cos(\lambda_N + \Omega t)\Big[1 -$$

$$+ \Big[(\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N\Big]_{(j+1)}\Big)\ \Big(\sin(\lambda_N + \Omega t)\Big[1 -$$

. . .

$$- \frac{(\Delta\lambda_N + \Omega\Delta t)^2}{6}\Bigg] - \sin(\lambda_N + \Omega t)\ \frac{\Delta\lambda_N + \Omega\Delta t}{2}\Bigg[1 -$$

$$- \frac{(\Delta\lambda_N + \Omega\Delta t)^2}{6}\Bigg] + \cos(\lambda_N + \Omega t)\ \frac{\Delta\lambda_N + \Omega\Delta t}{2}\Bigg[1 -$$

. . .

$$- \frac{(\Delta\lambda_N + \Omega\Delta t)^2}{12}\Bigg]\Bigg)$$

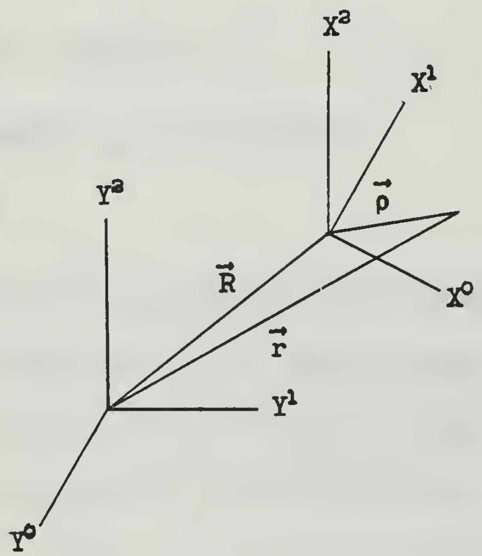
$$- \frac{(\Delta\lambda_N + \Omega\Delta t)^2}{12}\Bigg]\Bigg)\Bigg]$$

. . .

APPENDIX II

DEVELOPMENT OF FORMULAS FOR SENSORS ROTATING WITH THE EARTH

In the case where the accelerometers maintain a fixed orientation with respect to the earth, the X reference system is used, where X is identical to Y at time t = 0. Since the reference frame is rotating we must consider the effects of this rotation on the measured accelerations. The accelerometers continue to sense accelerations in the inertial frame, resolved into components along the X axes, rather than measuring accelerations relative to the X reference frame.



$\vec{r} = \vec{R} + \vec{\rho}$

$$\dot{\vec{r}}_Y = \dot{\vec{R}}_Y + \frac{d}{dt}(\vec{\rho})_Y$$

$$\dot{\vec{\rho}}_Y = \vec{\Omega}\times\vec{\rho} + \dot{\vec{\rho}}_X$$

$$\dot{\vec{r}}_Y = \dot{\vec{R}}_Y + \vec{\Omega}\times\vec{\rho} \quad \dot{\vec{\rho}}_X$$

$$\ddot{\vec{r}}_Y = \ddot{\vec{R}}_Y + \frac{d}{dt}(\vec{\Omega}\times\vec{\rho})_Y + \frac{d}{dt}(\dot{\vec{\rho}}_X)_Y$$

$$= \ddot{\vec{R}}_Y + \frac{d}{dt}(\vec{\Omega}\times\vec{\rho})_Y + \vec{\Omega}\times\dot{\vec{\rho}}_X + \ddot{\vec{\rho}}_X$$

$$\frac{d}{dt}(\vec{\Omega}\times\vec{\rho})_Y = \vec{\Omega}'\times\vec{\rho} + \vec{\Omega}\times\dot{\vec{\rho}}_Y$$

$$= \dot{\vec{\Omega}}\times\vec{\rho} + \vec{\Omega}\times(\vec{\Omega}\times\vec{\rho}) + \vec{\Omega}\times\dot{\vec{\rho}}_X$$

$$\ddot{\vec{r}}_Y = \ddot{\vec{R}}_Y + \dot{\vec{\Omega}}\times\vec{\rho} + \vec{\Omega}\times(\vec{\Omega}\times\vec{\rho}) + 2\vec{\Omega}\times\dot{\vec{\rho}}_X + \ddot{\vec{\rho}}_X = \vec{A}$$

In our case, where the X and Y systems have a common origin, and where the rotation rate of the X system is constant at Ω, we may write

$$\vec{A} = \vec{\Omega}\times(\vec{\Omega}\times\vec{\rho}) + 2\vec{\Omega}\times\dot{\vec{\rho}}_X + \ddot{\vec{\rho}}_X \qquad \text{II.1}$$

As in Appendix I, we may write

$$\vec{A}_s = \vec{A} - \vec{G} \qquad \text{II.2}$$

Although $\vec{A}$ is the acceleration in inertial space, it need not be sensed along the Y^0, Y^1 and Y^2 axes. In the present case we will sense along the X^0, X^1 and X^2 axes. The gravitation vector can be expressed in this system as:

$$\vec{G}_N = -\,(\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N \cos\lambda_N\,\vec{i}$$

$$-\,(\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N \sin\lambda_N\,\vec{j} \qquad \text{II.3}$$

$$-\,\gamma_N \sin\varphi_N\,\vec{k}$$

The expression for actual gravitation will be written in the same manner as was done in Appendix I, taking advantage of the fact that, for this study, the navigational experiment is being performed at or near an equipotential surface; therefore, gravity anomalies will be used rather than gravity disturbances.

$$\begin{aligned}\vec{G} = &-\Big[(\gamma + \Delta g)\cos(\varphi + \zeta)\cos(\lambda + \eta/\cos\varphi) + \\ &\qquad + \Omega^2 N\cos\varphi\cos\lambda\Big]\ \vec{i} \\ &-\Big[(\gamma + \Delta g)\cos(\varphi + \zeta)\sin(\lambda + \eta/\cos\varphi) + \\ &\qquad + \Omega^2 N\cos\varphi\sin\lambda\Big]\ \vec{j} \\ &-\Big[(\gamma + \Delta g)\sin(\varphi + \zeta)\Big]\ \vec{k}\end{aligned} \qquad \text{II.4}$$

Putting equations II.1 and II.3 into II.2, we write the equation that is used to solve the navigation problem from the sensed acceleration:

$$\begin{aligned}\vec{A}_s = &\Big[-\Omega^2 X_N^0 - 2\Omega\dot{X}_N^1 + \ddot{X}_N^0 + \\ &\qquad + (\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N\cos\lambda_N\Big]\ \vec{i} \\ &+\Big[-\Omega^2 X_N^1 + 2\Omega\dot{X}_N^0 + \ddot{X}_N^1 + \\ &\qquad + (\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N\sin\lambda_N\Big]\ \vec{j} \\ &+\Big[\ \ddot{X}_N^2 + \gamma_N\sin\varphi_N\Big]\ \vec{k}\end{aligned} \qquad \text{II.5}$$

To obtain the hypothetical expression for sensed acceleration, we write the same basic equation, but for the theoretical position, velocity and acceleration of the navigated instrument, and under the influence of hypothetical gravitation. To obtain this result, we put equations II.1 and II.4 into II.2 and obtain the results shown below.

$$\vec{A}_s = \Big[-\Omega^2 X^0 - 2\Omega\dot{X}^1 + \ddot{X}^0 + \Omega^2 N\cos\varphi\,\cos\lambda +$$
$$+ (\gamma + \Delta g)\cos(\varphi + \xi)\,\cos(\lambda + \eta/\cos\varphi)\Big]\,\vec{i} +$$
$$+\Big[-\Omega^2 X^1 + 2\Omega\dot{X}^0 + \ddot{X}^1 + \Omega^2 N\cos\varphi\,\sin\lambda + \qquad \text{II.6}$$
$$+ (\gamma + \Delta g)\cos(\varphi + \xi)\,\sin(\lambda + \eta/\cos\varphi)\Big]\,\vec{j} +$$
$$+\Big[\ddot{X}^2 + (\gamma + \Delta g)\sin(\varphi + \xi)\Big]\,\vec{k}$$

Setting $\dot{\vec{\rho}}_X = \ddot{\vec{\rho}}_X = 0$, and equating terms between II.5 and II.6, we obtain

$$\ddot{X}^0_N = -\Omega^2(X^0 - X^0_N) + 2\Omega\dot{X}^1_N + \Omega^2 N\cos\varphi\,\cos\lambda +$$
$$+ (\gamma + \Delta g)\cos(\varphi + \xi)\,\cos(\lambda + \eta/\cos\varphi) - \qquad \text{II.7a}$$
$$- (\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N\,\cos\lambda_N$$

$$\ddot{X}^1_N = -\Omega^2(X^1 - X^1_N) - 2\Omega\dot{X}^0_N + \Omega^2 N\cos\varphi\,\sin\lambda +$$
$$+ (\gamma + \Delta g)\cos(\varphi + \xi)\,\sin(\lambda + \eta/\cos\varphi) - \qquad \text{II.7b}$$
$$- (\gamma_N + \Omega^2(N_N + h_N))\cos\varphi_N\,\sin\lambda_N$$

$$\ddot{X}^2_N = (\gamma + \Delta g)\sin(\varphi + \xi) - \gamma_N\sin\varphi_N \qquad \text{II.7c}$$

In view of the fact that

$$X^0 = N\cos\varphi\,\cos\lambda \qquad X^0_N = (N_N + h_N)\cos\varphi_N\,\cos\lambda_N$$

$$X^1 = N\cos\varphi\,\sin\lambda \qquad X^1_N = (N_N + h_N)\cos\varphi_N\,\sin\lambda_N$$

equations II.7 reduce to the following expression.

$$\begin{bmatrix} \ddot{X}_N^0 \\ \ddot{X}_N^1 \\ \ddot{X}_N^2 \end{bmatrix} = 2\Omega \begin{bmatrix} \dot{X}_N^1 \\ -\dot{X}_N^0 \\ 0 \end{bmatrix} + (\gamma + \Delta g)\cos(\varphi + \xi) \begin{bmatrix} \cos(\lambda + \eta/\cos\varphi) \\ \sin(\lambda + \eta/\cos\varphi) \\ \tan(\varphi + \xi) \end{bmatrix} - \qquad \text{II.7'}$$

$$- \gamma_N \cos\varphi_N \begin{bmatrix} \cos\lambda_N \\ \sin\lambda_N \\ \tan\varphi_N \end{bmatrix}$$

Although the determination of the $\ddot{\vec{X}}_N$ components involves more computations than did the determination of the components of $\ddot{\vec{Y}}_N$, the integration which follows is still straightforward, and, since the gravitation vector for a point remains fixed with respect to the rotating coordinate axes (accelerometer axes), a simple mean may be used to obtain the average acceleration over a time increment in place of the more elaborate methods used in Appendix I.

$$\vec{\rho} = \vec{\rho}_0 + \dot{\vec{\rho}}_0 t + \int_0^t \int_0^v \ddot{\vec{\rho}} \, du \, dv$$

This equation may be written, incrementally, in a manner similar to that used in Appendix I, as

$$\vec{\rho}_{N(j+1)} = \vec{\rho}_{N(j)} + \dot{\vec{\rho}}_{N(j)}\Delta t + \frac{1}{4}(\Delta t)^2 \left[\ddot{\vec{\rho}}_{N(j)} + \ddot{\vec{\rho}}_{N(j+1)}\right]$$

$$
\left.
\begin{aligned}
\begin{bmatrix} X_N^0 \\ X_N^1 \\ X_N^2 \end{bmatrix}_{(j+1)} &= \begin{bmatrix} X_N^0 \\ X_N^1 \\ X_N^2 \end{bmatrix}_{(j)} + \begin{bmatrix} \mathring{X}_N^0 \\ \mathring{X}_N^1 \\ \mathring{X}_N^2 \end{bmatrix}_{(j)} \Delta t \; + \\
&+ \frac{1}{2}\Omega(\Delta t)^2 \begin{bmatrix} \mathring{X}_{N(j)}^1 + \mathring{X}_{N(j+1)}^1 \\ - \mathring{X}_{N(j)}^0 - \mathring{X}_{N(j+1)}^0 \\ 0 \end{bmatrix} \\
&+ \frac{1}{2}(\Delta t)^2(\gamma + \Delta g)\cos(\varphi + \xi) \begin{bmatrix} \cos(\lambda + \eta/\cos\varphi) \\ \sin(\lambda + \eta/\cos\varphi) \\ \tan(\varphi + \xi) \end{bmatrix} \\
&- \frac{1}{4}(\Delta t)^2 \gamma_{N(j)} \cos\varphi_{N(j)} \begin{bmatrix} \cos\lambda_N \\ \sin\lambda_N \\ \tan\varphi_N \end{bmatrix}_{(j)} \\
&- \frac{1}{4}(\Delta t)^2 \gamma_{N(j+1)} \cos\varphi_{N(j+1)} \begin{bmatrix} \cos\lambda_N \\ \sin\lambda_N \\ \tan\varphi_N \end{bmatrix}_{(j+1)}
\end{aligned}
\right\} \quad \text{II.8}
$$

APPENDIX III

DEVELOPMENT OF FORMULAS FOR SENSORS ORIENTED TO THE ELLIPSOIDAL NORMAL

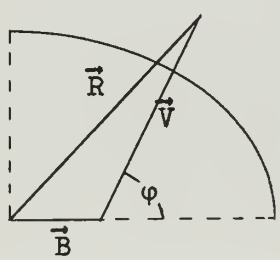
In the case where the accelerometers maintain an orientation with one axis parallel to the ellipsoidal normal, one axis in the horizon plane in the meridian, and the third axis in the horizon plane in the prime vertical, the Z coordinate system is used primarily; however, since the accelerations are sensed in inertial space, it is necessary to transform repeatedly between the Y system and the Z system. The transformation equations used are:

$$\vec{R}_Z = R_1(-\varphi)\, R_2(\Lambda)\, \vec{R}_Y = M_T \vec{R}_Y$$

$$M_T = \begin{bmatrix} +\cos\varphi \cos\Lambda & +\cos\varphi \sin\Lambda & +\sin\varphi \\ -\sin\Lambda & +\cos\Lambda & 0 \\ -\sin\varphi \cos\Lambda & -\sin\varphi \sin\Lambda & +\cos\varphi \end{bmatrix}$$

As before, $\vec{A}_S = \ddot{\vec{R}} - \vec{G}$. Let

$$\vec{R} = \vec{B} + \vec{V}$$



$$\vec{B}^o = \begin{bmatrix} \cos\Lambda \\ \sin\Lambda \\ 0 \end{bmatrix}$$

$$B = Ne^2 \cos\varphi$$

$$\vec{V}^o = \begin{bmatrix} \cos\varphi \ \cos\Lambda \\ \cos\varphi \ \sin\Lambda \\ \sin\varphi \end{bmatrix}$$

$$V = N(1-e^2) + h$$

$$\vec{R} = \vec{B}^o B + \vec{V}^o V \qquad \text{III.1}$$

$$\dot{\vec{R}} = \dot{\vec{B}}^o B + \vec{B}^o \dot{B} + \dot{\vec{V}}^o V + \vec{V}^o \dot{V}$$

$$= (\vec{\omega}_B \times \vec{B}^o)B + \vec{B}^o \dot{B} + (\vec{\omega}_V \times \vec{V}^o)V + \vec{V}^o \dot{V}$$

$$\ddot{\vec{R}} = (\dot{\vec{\omega}}_B \times \vec{B}^o)B + (\vec{\omega}_B \times \dot{\vec{B}}^o)B + (\vec{\omega}_B \times \vec{B}^o)\dot{B} + \dot{\vec{B}}^o B + \vec{B}^o \ddot{B} +$$

$$+ (\dot{\vec{\omega}}_V \times \vec{V}^o)V + (\vec{\omega}_V \times \dot{\vec{V}}^o)V + (\vec{\omega}_V \times \vec{V}^o)\dot{V} + \dot{\vec{V}}^o V + \vec{V}^o \ddot{V}$$

$$= (\dot{\vec{\omega}}_B \times \vec{B}^o)B + (\dot{\vec{\omega}}_V \times \vec{V}^o)V + \vec{\omega}_B \times (\vec{\omega}_B \times \vec{B}^o)B + \vec{\omega}_V \times (\vec{\omega}_V \times \vec{V}^o)V +$$

$$+ \vec{B}^o \ddot{B} + \vec{V}^o \ddot{V} + 2(\vec{\omega}_B \times \vec{B}^o)\dot{B} + 2(\vec{\omega}_V \times \vec{V}^o)\dot{V} \qquad \text{III.2}$$

$$M_T \vec{B}^o = \begin{bmatrix} \cos\varphi \\ 0 \\ -\sin\varphi \end{bmatrix}$$

$$M_T \vec{V}^o = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{B} = \left[\frac{Ne^4}{2W^2} \cos\varphi \sin2\varphi - Ne^2 \sin\varphi\right] \dot{\varphi}$$

$$= - \frac{Ne^2(1-e^2) \sin\varphi}{W^2} \dot{\varphi}$$

$$\ddot{B} = \left[- \frac{Ne^2}{2W^4} e^2(1-e^2)\sin\varphi \sin2\varphi - \frac{Ne^2(1-e^2)}{W^2} \cos\varphi\right.$$

$$\left. - \frac{Ne^4(1-e^2)}{W^4} \sin\varphi \sin2\varphi\right] \dot{\varphi}^2 - \frac{Ne^2(1-e^2)}{W^2} \sin\varphi \ddot{\varphi}$$

$$= \left[- \frac{3Ne^4}{2W^4} (1-e^2)\sin\varphi \sin2\varphi - \frac{Ne^2}{W^2} (1-e^2)\cos\varphi\right] \dot{\varphi}^2 -$$

$$- \frac{Ne^2(1-e^2)\sin\varphi}{W^2} \ddot{\varphi}$$

$$\dot{V} = \frac{Ne^2(1-e^2)}{2W^2} \sin2\varphi \dot{\varphi} + \dot{h}$$

$$\ddot{V} = \left[\frac{Ne^4(1-e^2)}{4W^4} \sin^2 2\varphi + \frac{Ne^4(1-e^2)}{2W^4} \sin^2 2\varphi + \frac{Ne^2(1-e^2)}{2W^2} 2 \cos2\varphi\right] \dot{\varphi}^2 +$$

$$+ \frac{Ne^2(1-e^2)}{2W^2} \sin2\varphi \ddot{\varphi} + \ddot{h}$$

$$= \left[\frac{3Ne^4(1-e^2)}{4W^4} \sin^2 2\varphi + \frac{Ne^2(1-e^2)}{W^2} \cos2\varphi\right] \dot{\varphi}^2 +$$

$$+ \frac{Ne^2(1-e^2)}{2W^2} \sin2\varphi \ddot{\varphi} + \ddot{h}$$

$$\vec{\omega}_B = \begin{bmatrix} 0 \\ 0 \\ \Omega + \dot{\lambda} \end{bmatrix}$$

$$\dot{\vec{\omega}}_B = \begin{bmatrix} 0 \\ 0 \\ \ddot{\lambda} \end{bmatrix}$$

$$\vec{\omega}_V = \begin{bmatrix} \dot{\varphi} \sin\Lambda \\ -\dot{\varphi} \cos\Lambda \\ \Omega + \dot{\lambda} \end{bmatrix}$$

$$\dot{\vec{\omega}}_V = \begin{bmatrix} \ddot{\varphi} \sin\Lambda + \dot{\varphi} \cos\Lambda \; (\Omega + \dot{\lambda}) \\ -\ddot{\varphi} \cos\Lambda + \dot{\varphi} \sin\Lambda \; (\Omega + \dot{\lambda}) \\ \ddot{\lambda} \end{bmatrix}$$

$$\left. M_T (\vec{B}^o \ddot{B} + \vec{V}^o \ddot{V}) = \begin{bmatrix} \ddot{h} - \frac{Ne^2 (1-e^2)}{W^3} \sin^2\varphi \; \dot{\varphi}^2 \\ 0 \\ \frac{Ne^2 (1-e^2)}{W^3} \frac{\sin 2\varphi}{2} \frac{3e^2}{W^2} \sin^2\varphi + 1 \quad \dot{\varphi}^2 \quad + \\ \cdots \\ \cdots \\ + \frac{Ne^2 (1-e^2)}{W^3} \sin^2\varphi \; \ddot{\varphi} \end{bmatrix} \right\} \quad \text{III.3}$$

$$\vec{\omega}_B \times \vec{B}^o = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \Omega + \dot{\lambda} \\ \cos\Lambda & \sin\Lambda & 0 \end{vmatrix} = \begin{bmatrix} -\sin\Lambda \; (\Omega + \dot{\lambda}) \\ \cos\Lambda \; (\Omega + \dot{\lambda}) \\ 0 \end{bmatrix}$$

$$\vec{\omega}_V \times \vec{V}^o = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{\varphi}\ \sin\Lambda & -\dot{\varphi}\ \cos\Lambda & \Omega + \dot{\lambda} \\ \cos\varphi\ \cos\Lambda & \cos\varphi\ \sin\Lambda & \sin\varphi \end{vmatrix} =$$

$$= \begin{bmatrix} -\sin\varphi\ \cos\Lambda\ \dot{\varphi} - \cos\varphi\ \sin\Lambda\ (\Omega + \dot{\lambda}) \\ -\sin\varphi\ \sin\Lambda\ \dot{\varphi} + \cos\varphi\ \cos\Lambda\ (\Omega + \dot{\lambda}) \\ \cos\varphi\ \dot{\varphi} \end{bmatrix}$$

$$M_T\,(\vec{\omega}_B \times \vec{B}^o) = \begin{bmatrix} 0 \\ \Omega + \dot{\lambda} \\ 0 \end{bmatrix}$$

$$M_T\,(\vec{\omega}_V \times \vec{V}^o) = \begin{bmatrix} 0 \\ \cos\varphi\ (\Omega + \dot{\lambda}) \\ \dot{\varphi} \end{bmatrix}$$

$$\left.\begin{aligned} & 2M_T \left[(\vec{\omega}_B \times \vec{B}^o)\dot{B} + (\vec{\omega}_V \times \vec{V}^o)\dot{V} \right] = \\ & = \begin{bmatrix} 0 \\ -2\,\dfrac{Ne^2(1-e^2)}{W^3} \sin^3\varphi\ \dot{\varphi}(\Omega + \dot{\lambda}) + 2\cos\varphi\ \dot{h}(\Omega + \dot{\lambda}) \\ 2\,\dfrac{Ne^2(1-e^2)}{W^3} \sin\varphi\ \cos\varphi\ \dot{\varphi}^2 + 2\,\dot{h}\dot{\varphi} \end{bmatrix} \end{aligned}\right\} \text{III.4}$$

$$\dot{\vec{\omega}}_B \times \vec{B}^o = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \ddot{\lambda} \\ \cos\Lambda & \sin\Lambda & 0 \end{vmatrix} = \begin{bmatrix} -\sin\Lambda\ \ddot{\lambda} \\ \cos\Lambda\ \ddot{\lambda} \\ 0 \end{bmatrix}$$

$$M_T(\dot{\vec{\omega}}_B \times \vec{B}^o) = \begin{bmatrix} 0 \\ \ddot{\lambda} \\ 0 \end{bmatrix}$$

$$\dot{\vec{\omega}}_V \times \vec{V}^o = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sin\Lambda\ \ddot{\varphi} + \cos\Lambda\ \dot{\varphi}(\Omega + \dot{\lambda}) & -\cos\Lambda\ \ddot{\varphi} + \sin\Lambda\ \dot{\varphi}(\Omega + \dot{\lambda}) & \ddot{\lambda} \\ \cos\varphi\ \cos\Lambda & \cos\varphi\ \sin\Lambda & \sin\varphi \end{vmatrix}$$

$$= \begin{bmatrix} -\sin\varphi\ \cos\Lambda\ \ddot{\varphi} + \sin\varphi\ \sin\Lambda\ \dot{\varphi}(\Omega + \dot{\lambda}) - \cos\varphi\ \sin\Lambda\ \ddot{\lambda} \\ -\sin\varphi\ \sin\Lambda\ \ddot{\varphi} - \sin\varphi\ \cos\Lambda\ \dot{\varphi}(\Omega + \dot{\lambda}) + \cos\varphi\ \cos\Lambda\ \ddot{\lambda} \\ +\cos\varphi\ \ddot{\varphi} \end{bmatrix}$$

$$M_T(\dot{\vec{\omega}}_V \times \vec{V}^o) = \begin{bmatrix} 0 \\ -\sin\varphi\ \dot{\varphi}(\Omega + \dot{\lambda}) + \cos\varphi\ \ddot{\lambda} \\ \ddot{\varphi} \end{bmatrix}$$

$$
\left.\begin{aligned}
&M_T\left[(\dot{\vec{\omega}}_B \times \vec{B}^o)B + (\dot{\vec{\omega}}_V \times \vec{V}^o)V\right] = \\
&= \begin{bmatrix} 0 \\ (N + h)\cos\varphi\,\ddot{\lambda} - (N(1-e^2) + h)\sin\varphi\,\dot{\varphi}(\Omega + \dot{\lambda}) \\ (N(1-e^2) + h)\,\ddot{\varphi} \end{bmatrix}
\end{aligned}\right\} \text{III.5}
$$

$$
\vec{\omega}_B \times (\vec{\omega}_B \times \vec{B}^o) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \Omega + \dot{\lambda} \\ -\sin\Lambda\,(\Omega + \dot{\lambda}) & \cos\Lambda\,(\Omega + \dot{\lambda}) & 0 \end{vmatrix}
$$

$$
= \begin{bmatrix} -\cos\Lambda\,(\Omega + \dot{\lambda})^2 \\ -\sin\Lambda\,(\Omega + \dot{\lambda})^2 \\ 0 \end{bmatrix}
$$

$$
M_T(\vec{\omega}_B \times (\vec{\omega}_B \times \vec{B}^o)) = \begin{bmatrix} -\cos\varphi\,(\Omega + \dot{\lambda})^2 \\ 0 \\ \sin\varphi\,(\Omega + \dot{\lambda})^2 \end{bmatrix}
$$

$$
\vec{\omega}_V \times (\vec{\omega}_V \times \vec{V}^o) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{\varphi}\sin\Lambda & -\dot{\varphi}\cos\Lambda & \Omega + \dot{\lambda} \\ \begin{bmatrix} -\sin\varphi\cos\Lambda\,\dot{\varphi} - \\ -\cos\varphi\sin\Lambda\,(\Omega + \dot{\lambda}) \end{bmatrix} & \begin{bmatrix} -\sin\varphi\sin\Lambda\,\dot{\varphi} + \\ +\cos\varphi\cos\Lambda\,(\Omega + \dot{\lambda}) \end{bmatrix} & \cos\varphi\,\dot{\varphi} \end{vmatrix}
$$

$$\vec{\omega}_V \times (\vec{\omega}_V \times \vec{V}^o) = \begin{bmatrix} -\cos\varphi\cos\Lambda\,\dot{\varphi}^2 + \sin\varphi\sin\Lambda\,\dot{\varphi}(\Omega + \dot{\lambda}) - \\ -\cos\varphi\sin\Lambda\,\dot{\varphi}^2 - \sin\varphi\cos\Lambda\,\dot{\varphi}(\Omega + \dot{\lambda}) - \\ -\sin\varphi\,\dot{\varphi}^2 \end{bmatrix}$$

$$\begin{matrix} -\cos\varphi\cos\Lambda\,(\Omega + \dot{\lambda})^2 \\ -\cos\varphi\sin\Lambda\,(\Omega + \dot{\lambda})^2 \\ \cdots \end{matrix}\Bigg]$$

$$M_T(\vec{\omega}_V \times (\vec{\omega}_V \times \vec{V}^o)) = \begin{bmatrix} -\dot{\varphi}^2 - \cos^2\varphi\,(\Omega + \dot{\lambda})^2 \\ -\sin\varphi\,\dot{\varphi}(\Omega + \dot{\lambda}) \\ +\sin\varphi\cos\varphi\,(\Omega + \dot{\lambda})^2 \end{bmatrix}$$

$$\left.\begin{aligned} & M_T\left[\vec{\omega}_B \times (\vec{\omega}_B \times \vec{B}^o)B + \vec{\omega}_V \times (\vec{\omega}_V \times \vec{V}^o)V\right] = \\ & = \begin{bmatrix} -(N(1-e^2) + h)\,\dot{\varphi}^2 - (N + h)\cos^2\varphi\,(\Omega + \dot{\lambda})^2 \\ -(N(1-e^2) + h)\sin\varphi\,\dot{\varphi}(\Omega + \dot{\lambda}) \\ +(N + h)\sin\varphi\cos\varphi\,(\Omega + \dot{\lambda})^2 \end{bmatrix} \end{aligned}\right\} \quad \text{III.6}$$

Equations III.3 through III.6 provide the components of $\ddot{\vec{R}}$ that go into equation III.2. To determine $\ddot{\vec{R}}$ as a function of the sensed acceleration, it is necessary to obtain the components of the gravitation vector also. From considerations similar to those of Appendix I, we will derive the gravitation vector from the

gravity vector, since we are considering navigation on or near the reference ellipsoid in the ideal case.

$$- \vec{G} + \vec{\Omega} \times (\vec{\Omega} \times (N + h)\vec{V}^{o}) = - \vec{\gamma}$$

$$M_T(-\vec{G}) = \begin{bmatrix} \gamma + (N + h)\cos^2\varphi \, \Omega^2 \\ 0 \\ -(N + h)\sin\varphi \cos\varphi \, \Omega^2 \end{bmatrix} \qquad \text{III.7}$$

$$\vec{A}_s = \begin{bmatrix} \ddot{Z}_s^0 \\ \ddot{Z}_s^1 \\ \ddot{Z}_s^2 \end{bmatrix} = \begin{bmatrix} \ddot{h} - \left[\frac{N(1-e^2)}{W^2} + h\right]\dot{\varphi}^2 - (N + h)\cos^2\varphi \, \dot{\lambda}(2\Omega + \dot{\lambda}) + \gamma \\ (N + h)\cos\varphi \, \ddot{\lambda} - 2\left[\frac{N(1-e^2)}{W^2} + h\right] \sin\varphi \, \dot{\varphi}(\Omega + \dot{\lambda}) + \\ \left[\frac{N(1-e^2)}{W^2} + h\right] \ddot{\varphi} + \frac{3N(1-e^2)}{2W^2}\frac{e^2}{W^2} \sin 2\varphi \, \dot{\varphi}^2 + 2\dot{h}\dot{\varphi} + \end{bmatrix}$$

$$\begin{matrix} \cdots \\ + 2\cos\varphi \, \dot{h}(\Omega + \dot{\lambda}) \\ + (N + h)\sin\varphi \cos\varphi \, \dot{\lambda}(2\Omega + \dot{\lambda}) \end{matrix} \Bigg]$$

In view of the fact that $\frac{N(1-e^2)}{W^2}$ is the radius of curvature of the ellipsoid in the meridian, M, we may rewrite the equations for the sensed accelerations in the more compact form shown below.

$$\begin{bmatrix} \ddot{Z}_s^0 \\ \ddot{Z}_s^1 \\ \ddot{Z}_s^2 \end{bmatrix} = \begin{bmatrix} \ddot{h} - (M + h)\dot{\varphi}^2 - (N + h)\cos^2\varphi\, \dot{\lambda}(2\Omega + \dot{\lambda}) + \gamma \\ (N + h)\cos\varphi\, \ddot{\lambda} - 2(M + h)\sin\varphi\, \dot{\varphi}(\Omega + \dot{\lambda}) + 2\cos\phi\, \dot{h}(\Omega + \dot{\lambda}) \\ (M + h)\ddot{\varphi} + 3\,\frac{Me^2}{W^2}\sin\varphi\cos\varphi\, \dot{\varphi}^2 + 2\dot{h}\dot{\varphi} + (N + h)\sin\varphi\cos\varphi\, \dot{\lambda}(2\Omega + \dot{\lambda}) \end{bmatrix} \qquad \text{III.8}$$

The above equations, when evaluated at the navigator's position, permit the determination of $\ddot{h}$, $\ddot{\lambda}$ and $\ddot{\varphi}$ from the sensed accelerations, under the assumption of normal gravity. These equations are equivalent to the kinematic equations of motion presented in standard works on inertial navigation, but these equations are in terms of fundamental parameters used by geodesists (M, N, e^2, φ, λ, h) rather than in terms which have more general applicability, and which are used to fit any situation (radius, geocentric latitude, longitude), but which must then be transformed to obtain the quantities which the navigator desires.

As in Appendixes I and II, a problem will be considered in which a navigation system is used which is initially in perfect alignment with the coordinate system at the geodetic position of the observer, but in which the gravity vector does not coincide with normal gravity. The centrifugal effects given in equation

III.7 will remain unchanged, since they are a function of position only, but the net gravity effect will change to

$$(\vec{g})_z = R_1(-\varphi)R_2(\Lambda)R_2(-\Lambda - \eta/\cos\varphi)R_1(\varphi + \xi)\begin{bmatrix}\gamma + \Delta g \\ 0 \\ 0\end{bmatrix}$$

Using this relationship, and tacitly assuming that we have no geoidal undulation with which to contend, it is possible to rewrite equation III.8 for the sensed accelerations in the hypothetically true case as

$$\left.\begin{aligned}
\begin{bmatrix}\ddot{Z}_s^0 \\ \ddot{Z}_s^1 \\ \ddot{Z}_s^2\end{bmatrix} = &\begin{bmatrix}\ddot{h} - (M + h)\dot{\varphi}^2 - (N + h)\cos^2\varphi\ \dot{\lambda}(2\Omega + \dot{\lambda}) \\ (N + h)\cos\varphi\ \ddot{\lambda} - 2(M + h)\sin\varphi\ \dot{\varphi}(\Omega + \dot{\lambda}) + \\ (M + h)\ddot{\varphi} + 3\dfrac{Me^2}{W^2}\sin\varphi\cos\varphi\ \dot{\varphi}^2 + 2\dot{h}\dot{\varphi} + \end{bmatrix} \\
&\quad \begin{matrix} \cdots \\ + 2\cos\varphi\ \dot{h}(\Omega + \dot{\lambda}) \\ + (N + h)\sin\varphi\cos\varphi\ \dot{\lambda}(2\Omega + \dot{\lambda}) \end{matrix}\Bigg] + \\
&+ (\gamma + \Delta g)\begin{bmatrix}\sin\varphi\sin(\varphi + \xi) + \\ \cdots - \\ \cos\varphi\sin(\varphi + \xi) - \end{bmatrix}
\end{aligned}\right\}\ \text{III.9}$$

$$
\begin{aligned}
&+ \cos\varphi \cos(\varphi + \zeta) \cos\Lambda \cos(\Lambda + \eta/\cos\varphi) + \\
&- \cos(\varphi + \zeta) \sin\Lambda \cos(\Lambda + \eta/\cos\varphi) + \\
&- \sin\varphi \cos(\varphi + \zeta) \cos\Lambda \cos(\Lambda + \eta/\cos\varphi) - \\
&\left.\begin{aligned}
&+ \cos\varphi \cos(\varphi + \zeta) \sin\Lambda \sin(\Lambda + \eta/\cos\varphi) \\
&+ \cos(\varphi + \zeta) \cos\Lambda \sin(\Lambda + \eta/\cos\varphi) \\
&- \sin\varphi \cos(\varphi + \zeta) \sin\Lambda \sin(\Lambda + \eta/\cos\varphi)
\end{aligned}\right]
\end{aligned}
\Bigg)
$$

In the cases which have been considered in the first two appendixes, the navigator has been stationary with respect to the earth, so that $\ddot{h} = \ddot{\varphi} = \ddot{\lambda} = \dot{h} = \dot{\varphi} = \dot{\lambda} = 0$. In this situation equation III.9 simplifies to

$$
\begin{bmatrix} \ddot{Z}_s^0 \\ \ddot{Z}_s^1 \\ \ddot{Z}_s^2 \end{bmatrix} = (\gamma + \Delta g)\cos(\varphi + \zeta)\, M_\tau \begin{bmatrix} \cos(\Lambda + \eta/\cos\varphi) \\ \sin(\Lambda + \eta/\cos\varphi) \\ \tan(\varphi + \zeta) \end{bmatrix} \qquad \text{III.10}
$$

If the stable platform were to maintain its orientation with respect to the normal to the ellipsoid, and to rotate such that the Z^2 axis remained aligned in the meridian, the above quantity would remain constant. Unfortunately, the rate of driving the platform about the various axes is a function both of indicated velocity and indicated latitude, and the axes themselves are constantly moving, making the problem difficult. The rotation rate about the instantaneous Z^1 axis, which, for the navigator, is

defined as east regardless of its actual orientation, is $\dot{\varphi}_N$. The rate about the instantaneous Z^3 axis is $(\Omega + \dot{\lambda}_N)\cos\varphi_N$, and that about the instantaneous Z^0 axis is $(\Omega + \dot{\lambda}_N)\sin\varphi_N$. Since these rotations are not vector quantities, the order in which each rotation is accomplished is important; however, these rotations are accomplished simultaneously in practice and in infinitesimal increments.

Let us consider that during one time increment, dt, the transformation matrix, M_T, is modified by the transformation

$$M_{T\,(j+1)} = R_0((\Omega + \dot{\lambda}_N)\sin\varphi_N\ dt)R_1(-\dot{\varphi}_N\ dt)R_3((\Omega + \dot{\lambda}_N)\cos\varphi_N\ dt)M_{T\,(j)}$$

This expression would become quite unwieldy when expanded; however, if we limit dt such that we may consider the cosines of the resultant angles to be one, the sines as the angles in radians, and the product of two angles to be negligible, the order of multiplication becomes unimportant, and the relationship may be approximated by

$$M_{T\,(j+1)} = M_\varepsilon M_{T\,(j)}$$

where

$$M_\varepsilon = \begin{bmatrix} +1 & +\dot{\Lambda}_N\cos\varphi_N\ dt & \dot{\varphi}_N\ dt \\ -\dot{\Lambda}_N\cos\varphi_N\ dt & +1 & \dot{\Lambda}_N\sin\varphi_N\ dt \\ -\dot{\varphi}_N\ dt & -\dot{\Lambda}_N\sin\varphi_N\ dt & 1 \end{bmatrix}$$

where $\dot{\Lambda}_N = \Omega + \dot{\lambda}_N$. At any time t we may write

$$\begin{bmatrix} \ddot{Z}_s^0 \\ \ddot{Z}_s^1 \\ \ddot{Z}_s^2 \end{bmatrix} = (\gamma + \Delta g) \prod_{v=0}^{t} M_\varepsilon(v) M_{T(0)} \cos(\varphi + \zeta) \begin{bmatrix} \cos(\Lambda + \eta/\cos\varphi) \\ \sin(\Lambda + \eta/\cos\varphi) \\ \tan(\varphi + \zeta) \end{bmatrix} \qquad \text{III.11}$$

The evaluation of equation III.11 can be extremely tedious because of the small time step required in the evaluation of the matrix $M_\varepsilon(v)$. We may rewrite that equation in terms of individual rotation matrices in the vollowing manner:

$$\ddot{\vec{R}}_{Z\,s} = \prod_i \Big[R_1(-\dot{\varphi}_{N_i} dt_i) R_0((\Omega + \dot{\lambda}_{N_i}) \sin\varphi_{N_i} dt_i) R_2((\Omega + \dot{\lambda}_{N_i}) \cos\varphi_{N_i} dt_i) \Big] R_1(-\varphi) R_2(\lambda) \times \prod_j \Big[R_2(-\Omega dt_j) \Big] \vec{g}_X \qquad \text{III.12}$$

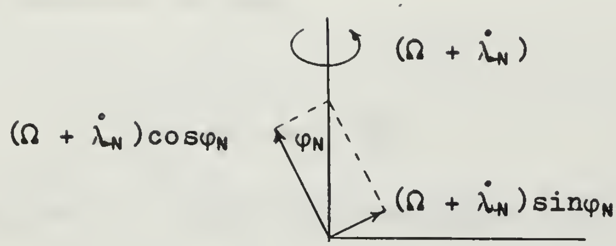
where the product $R_2(-\Omega dt_j)$ transforms the gravity vector from X space to inertial (Y) space. The product $R_1(-\varphi)R_2(\lambda)$ transforms it to the Z space as defined at time t = 0, and the long product which makes up the remainder of the expression transforms to Z space as defined at time t, which corresponds to the instantaneous sensor axes.

The two rotation matrices

$$R_0((\Omega + \dot{\lambda}_N)\sin\varphi_N \; dt) R_2((\Omega + \dot{\lambda}_N)\cos\varphi_N \; dt)$$

represent nothing but the component rotations of $(\Omega + \dot{\lambda}_N)$ about the vertical and north axes. These can be replaced by a matrix

$R_2((\Omega + \dot{\lambda}_N)dt)$ placed where the axis of rotation is either X^3 or Y^3.



Thus, equation III.12 may be written

$$\ddot{\vec{R}}_{z\,3} = \prod_i \left[R_1(-\dot{\varphi}_{N_i} dt_i)\right] R_1(-\varphi)R_3(\lambda) \times$$

$$\times \prod_j \left[R_3((\Omega + \dot{\lambda}_{N_j})dt_j)R_3(-\Omega dt_j)\right] \vec{g}_x$$

$$= \prod_i \left[R_1(-\dot{\varphi}_{N_i} dt_i)\right] R_1(-\varphi)R_3(\lambda) \prod_j \left[R_3(\dot{\lambda}_{N_j} dt_j)\right] \vec{g}_x \qquad \text{III.13}$$

Since the product of rotation matrices about the same axis is merely the rotation matrix for the sum of the individual arguments, equation III.13 may be written as

$$\ddot{\vec{R}}_{z\,3} = R_1\left[\int_0^t -\dot{\varphi}_N dt\right] R_1(-\varphi)R_3(\lambda)R_3\left[\int_0^t \dot{\lambda}_N dt\right] \vec{g}_x$$

$$= R_1(-\varphi_N)R_3(\lambda_N)(\gamma + \Delta g)\cos(\varphi + \xi)\begin{bmatrix} \cos(\lambda + \eta/\cos\varphi) \\ \sin(\lambda + \eta/\cos\varphi) \\ \tan(\varphi + \xi) \end{bmatrix} \qquad \text{III.14}$$

Combining equations III.8 and III.14, we obtain expressions for the observed accelerations of the navigator in height, longitude and latitude as follows.

$$\left.\begin{aligned}
\begin{bmatrix} \ddot{h}_N \\ (N+h)_N\cos\varphi_N\ \ddot{\lambda}_N \\ (M+h)_N\ddot{\varphi}_N \end{bmatrix} &= \begin{bmatrix} +\ (M+h)_N\ \dot{\varphi}_N^2\ + \\ +\ 2(M+h)_N\sin\varphi_N\ \dot{\varphi}_N(\Omega+\dot{\lambda}_N)\ - \\ -\ 3M_N e^2\sin 2\varphi_N\ \dot{\varphi}_N^2/2W_N^2\ - \end{bmatrix. \\
&\begin{matrix} +\ (N+h)_N\cos^2\varphi_N\ \dot{\lambda}_N(2\Omega+\dot{\lambda}_N)\ -\ \gamma_N \\ -\ 2\cos\varphi_N\ \dot{h}_N(\Omega+\dot{\lambda}_N) \\ -\ 2\dot{h}_N\dot{\varphi}_N\ -\ (N+h)_N\sin\varphi_N\ \cos\varphi_N\ \dot{\lambda}_N(2\Omega+\dot{\lambda}_N) \end{matrix}\Bigg] + \\
&+\ R_1(-\varphi_N)R_3(\lambda_N)(\gamma+\Delta g)\cos(\varphi+\xi)\begin{bmatrix}\cos(\lambda+\eta/\cos\varphi) \\ \sin(\lambda+\eta/\cos\varphi) \\ \tan(\varphi+\xi)\end{bmatrix}
\end{aligned}\right\rbrace \text{III.15}$$

$$\ddot{\lambda}_{N\ av} = \frac{1}{2}(\gamma+\Delta g)\cos(\varphi+\xi)\ \text{x}$$

$$\left[\left[\frac{-\ \sin\lambda_N\ \cos(\lambda+\eta/\cos\varphi)\ +\ \cos\lambda_N\ \sin(\lambda+\eta/\cos\varphi)}{(N_N+h_N)\cos\varphi_N}\right]_{(j)} +\right.$$

$$\left. +\left[\frac{-\ \sin\lambda_N\ \cos(\lambda+\eta/\cos\varphi)\ +\ \cos\lambda_N\ \sin(\lambda+\eta/\cos\varphi)}{(N_N+h_N)\cos\varphi_N}\right]_{(j+1)}\right] +$$

$$+\left[\frac{(M_N + h_N)\sin\varphi_N\ \mathring{\varphi}_N(\Omega + \mathring{\lambda}_N) - \cos\varphi_N\,\mathring{h}_N(\Omega + \mathring{\lambda}_N)}{(N_N + h_N)\cos\varphi_N}\right]_{(j)} +$$

$$+\left[\frac{(M_N + h_N)\sin\varphi_N\ \mathring{\varphi}_N(\Omega + \mathring{\lambda}_N) - \cos\varphi_N\ \mathring{h}_N(\Omega + \mathring{\lambda}_N)}{(N_N + h_N)\cos\varphi_N}\right]_{(j+1)}$$

Collecting terms, and putting $h_N = \mathring{h}_N = 0$, we obtain

$$\left.\begin{aligned}
\ddot{\lambda}_{N\,av} = \frac{-(\gamma + \Delta g)\cos(\varphi + \xi)}{2}\Bigg[\left[\frac{\sin(\lambda_N - \lambda - \eta/\cos\varphi)}{N_N\cos\varphi_N}\right]_{(j)} + \\
+\left[\frac{\sin(\lambda_N - \lambda - \eta/\cos\varphi)}{N_N\cos\varphi_N}\right]_{(j+1)}\Bigg] + \left[\frac{M_N\sin\varphi_N\mathring{\varphi}_N(\Omega + \mathring{\lambda}_N)}{N_N\cos\varphi_N}\right]_{(j)} + \\
+\left[\frac{M_N\sin\varphi_N\ \mathring{\varphi}_N(\Omega + \mathring{\lambda}_N)}{N_N\cos\varphi_N}\right]_{(j+1)}
\end{aligned}\right\} \text{III.16}$$

$$\begin{aligned}
\ddot{\varphi}_{N\,av} = \frac{\gamma + \Delta g}{2(M_N + h_N)_{(j)}}\Big[&- \sin\varphi_N\ \cos\lambda_N\ \cos(\varphi + \xi)\ \cos(\lambda + \eta/\cos\varphi) - \\
&- \sin\varphi_N\ \sin\lambda_N\ \cos(\varphi + \xi)\ \sin(\lambda + \eta/\cos\varphi) + \\
&+ \cos\varphi_N\ \sin(\varphi + \xi)\Big]_{(j)} + \\
+ \frac{\gamma + \Delta g}{2(M_N + h_N)_{(j+1)}}\Big[&- \sin\varphi_N\ \cos\lambda_N\ \cos(\varphi + \xi)\ \cos(\lambda + \eta/\cos\varphi) - \\
&- \sin\varphi_N\ \sin\lambda_N\ \cos(\varphi + \xi)\ \sin(\lambda + \eta/\cos\varphi) + \\
&+ \cos\varphi_N\ \sin(\varphi + \xi)\Big]_{(j+1)} - \\
- \frac{3e^2}{4(M_N + h_N)_{(j)}}&\left[M_N\sin 2\varphi_N\ \mathring{\varphi}_N^2/W_N^2\right]_{(j)} - \left[\frac{\mathring{h}_N\mathring{\varphi}_N}{M_N + h_N}\right]_{(j)} -
\end{aligned}$$

$$-\left[\frac{N_N + h_N}{4(M_N + h_N)} \sin 2\varphi_N \; \mathring{\lambda}_N (2\Omega + \mathring{\lambda}_N)\right]_{(j)} -$$

$$- \frac{3e^2}{4(M_N + h_N)_{(j+1)}} \left[M_N \sin 2\varphi_N \; \mathring{\varphi}_N^2 / W_N^2\right]_{(j+1)} - \left[\frac{\mathring{h}_N \mathring{\varphi}_N}{M_N + h_N}\right]_{(j+1)} -$$

$$-\left[\frac{N_N + h_N}{4(M_N + h_N)} \sin 2\varphi_N \; \mathring{\lambda}_N (2\Omega + \mathring{\lambda}_N)\right]_{(j+1)}$$

Collecting terms, and putting $h_N = \mathring{h}_N = 0$, we have

$$\left.\begin{aligned}
\ddot{\varphi}_{N\,av} = {} & \frac{\gamma + \Delta g}{2M_N}_{(j)} \Big[- \sin\varphi_N \cos(\varphi + \xi) \cos(\lambda_N - \lambda - \eta/\cos\varphi) + \\
& + \cos\varphi_N \sin(\varphi + \xi)\Big]_{(j)} + \\
& + \frac{\gamma + \Delta g}{2M_N}_{(j+1)} \Big[- \sin\varphi_N \cos(\varphi + \xi) \cos(\lambda_N - \lambda - \eta/\cos\varphi) + \\
& + \cos\varphi_N \sin(\varphi + \xi)\Big]_{(j+1)} - \\
& - \frac{3e^2}{4}\left[\sin 2\varphi_N \; \mathring{\varphi}_N^2 / W_N^2\right]_{(j)} - \left[\frac{N_N}{4M_N} \sin 2\varphi_N \; \mathring{\lambda}_N (2\Omega + \mathring{\lambda}_N)\right]_{(j)} - \\
& - \frac{3e^2}{4}\left[\sin 2\varphi_N \; \mathring{\varphi}_N^2 / W_N^2\right]_{(j+1)} - \left[\frac{N_N}{4M_N} \sin 2\varphi_N \; \mathring{\lambda}_N (2\Omega + \mathring{\lambda}_N)\right]_{(j+1)}
\end{aligned}\right\} \text{III.17}$$

Once the average accelerations in latitude and longitude are computed, the determination of position follows as

$$\lambda_{(j+1)} = \lambda_{(j)} + \mathring{\lambda}_{(j)}\Delta t + \frac{1}{2}(\Delta t)^2 \; \ddot{\lambda}_{av}$$

$$\varphi_{(j+1)} = \varphi_{(j)} + \mathring{\varphi}_{(j)}\Delta t + \frac{1}{2}(\Delta t)^2 \; \ddot{\varphi}_{av}$$

The development in the appendix through equation III.15 would indicate that computation for height would be possible in

addition to the computation for latitude and longitude. The development was not completed since, as the results of the two previous sensor configurations show, a solution for height in an imperfectly known gravitational field leads to ridiculous results. One of the primary objects of the present orientation of the sensors is to isolate the latitude and longitude sensors from the height sensor, recognizing that it is necessary to supplement the inertial data with information from another source for height information.

APPENDIX IV

DEVELOPMENT OF FORMULAS FOR NAVIGATION ON A HYPOTHETICALLY UNDULATING SURFACE

In the first three appendixes we have considered the effects of anomalous gravity on a navigator which was stationary with respect to the surface of the earth. Referring to equation III.9 we see that by having the navigator stationary we have eliminated a large part of the equation, which could, in fact, provide a significant contribution to the sensed value of acceleration. In this appendix we shall consider the more realistic case of a vessel moving over the surface of the earth, wherein it is subject to a varying effect of gravity, and where all the factors which contribute to the sensed value of acceleration must be considered.

Let us consider the case where the geoidal undulations may be expressed by

$$\zeta = \zeta_0 \left[\cos \frac{2\pi(\lambda - \lambda_0)}{(\lambda_1 - \lambda_0)} - 1\right]\left[\cos \frac{2\pi(\varphi - \varphi_0)}{(\varphi_1 - \varphi_0)} - 1\right] \qquad \text{IV.1}$$

where λ_0, λ_1, φ_0, and φ_1 are the boundaries of the tessera in which undulations are to be computed and $4\zeta_0$ is the value of the undulation at the center of the area. Using the relationships

$$\xi = -\frac{1}{R}\frac{\partial\zeta}{\partial\varphi} \qquad \text{and} \qquad \eta = -\frac{1}{R\cos\varphi}\frac{\partial\zeta}{\partial\lambda}$$

we may write analytical expressions for the components of the deflection of the vertical in the following form.

$$\xi = \frac{\zeta_0}{R}\left[\cos\frac{2\pi(\lambda - \lambda_0)}{(\lambda_1 - \lambda_0)} - 1\right]\frac{2\pi}{\varphi_1 - \varphi_0}\sin\frac{2\pi(\varphi - \varphi_0)}{(\varphi_1 - \varphi_0)} \qquad \text{IV.2}$$

$$\eta = \frac{\zeta_0}{R\cos\varphi}\left[\cos\frac{2\pi(\varphi - \varphi_0)}{(\varphi_1 - \varphi_0)} - 1\right]\frac{2\pi}{\lambda_1 - \lambda_0}\sin\frac{2\pi(\lambda - \lambda_0)}{(\lambda_1 - \lambda_0)} \qquad \text{IV.3}$$

Although ξ and η are directly dependent on the variations of the undulation, ζ, there is no such rigid dependency between ζ and the gravity anomalies. We may start from the principal equation of gravimetric geodesy

$$\Delta g = - 2\zeta\frac{\gamma}{R} - \frac{\partial T}{\partial R}$$

and see that since T, the disturbing potential, has not been specified in a form with a unique derivative, we are relatively free in our choice of the hypothetical distribution of anomalies. The tests described in Chapter II seem to indicate that the magnitude of gravity is relatively unimportant, so long as we have an independent means of measuring height. As a check on the importance of Δg, we shall let it vary in two different ways, then compare the results. In one case we shall assume that Δg in milligals equals ζ in decimeters, while in the other case we shall assume the negative of this relationship to hold.

Let us consider the instant of time at which we are steaming at a speed V, and at an azimuth α. The relationships which relate the rates of change of latitude, longitude and elevation

to velocity and azimuth on the surface are

$$\mathring{\varphi} = \frac{V\cos(\xi^2 + \eta^2)^{1/2}\cos\alpha}{M + H + \zeta}$$

$$\mathring{\lambda} = \frac{V\cos(\xi^2 + \eta^2)^{1/2}\sin\alpha}{(N + H + \zeta)\cos\varphi}$$

$$\mathring{h} = V\sin(\xi^2 + \eta^2)^{1/2}$$

Since, for this problem in navigation, M, N, ξ, η, ζ, φ, V, H and α are functions of time, the analytic expressions for $\ddot{\varphi}$, $\ddot{\lambda}$ and $\ddot{h}$, which are needed in equation III.9, get quite lengthy. In order to avoid the use of a very small time increment, we will establish the simplifying condition that V, H and α are constant; however, the fact that H is constant does not imply that $\mathring{h}$ is zero, since $\mathring{h}$ must take care of the variations in ζ. Since the establishment of analytic functions for $\ddot{\varphi}$, $\ddot{\lambda}$ and $\ddot{h}$ is a formidable task, even with the simplifying conditions, we will use numerical methods to arrive at these quantities.

We know that $\mathring{\varphi}$, $\mathring{\lambda}$ and $\mathring{h}$ are well-behaved, slowly changing functions. Using standard prediction techniques, we will fit a curve to the functions at the four most recent time points, then use this curve to predict the value of the function at the time point next to be considered, and the point immediately thereafter. The difference in the function at $t_{(j+1)}$ and $t_{(j-1)}$, divided by the time increment between the two points will give the value, to an acceptable order of accuracy, of the time derivative of the

function at $t_{(j)}$. The imposition of an iterative cycle if rigid limits on the difference between predicted and computed positions and functions are not met insures that we will not use values which are significantly in error in any of our equations.

As a set of equations equivalent to equations III.10 and III.8, we may write

$$
\left.
\begin{aligned}
&\begin{bmatrix} \ddot{Z}_s^0 \\ \ddot{Z}_s^1 \\ \ddot{Z}_s^2 \end{bmatrix} = (\gamma + \Delta g)\cos(\varphi + \xi)\ M_T \begin{bmatrix} \cos(\lambda + \eta/\cos\varphi + \Omega t) \\ \sin(\lambda + \eta/\cos\varphi + \Omega t) \\ \tan(\varphi + \xi) \end{bmatrix} + \\
&+ \begin{bmatrix} \ddot{h} - (M + H + \zeta)\mathring{\varphi}^2 - (N + H + \zeta)\cos^2\varphi\ \mathring{\lambda}(2\Omega + \mathring{\lambda}) \\ (N + H + \zeta)\cos\varphi\ \ddot{\lambda} - 2(M + H + \zeta)\sin\varphi\ \mathring{\varphi}(\Omega + \mathring{\lambda}) + \\ (M + H + \zeta)\ddot{\varphi} + 3Me^2\sin\varphi\cos\varphi\ \mathring{\varphi}^2/W^2 + 2\mathring{h}\mathring{\varphi} + \end{bmatrix} \\
&\begin{bmatrix} \circ\ \circ\ \circ \\ + 2\cos\varphi\ \mathring{h}(\Omega + \mathring{\lambda}) \\ + (N + H + \zeta)\sin\varphi\cos\varphi\ \mathring{\lambda}(2\Omega + \mathring{\lambda}) \end{bmatrix} = \\
&= \begin{bmatrix} \ddot{h}_N - (M_N + h_N)\mathring{\varphi}_N^2 - (N_N + h_N)\cos^2\varphi_N\ \mathring{\lambda}_N(2\Omega + \mathring{\lambda}_N) + \\ (N_N + h_N)\cos\varphi_N\ \ddot{\lambda}_N - 2(M_N + h_N)\sin\varphi_N\ \mathring{\varphi}_N(\Omega + \mathring{\lambda}_N) + \\ (M_N + h_N)\ddot{\varphi}_N + 3M_N e^2\sin\varphi_N\cos\varphi_N\ \mathring{\varphi}_N^2/W_N^2 + 2\mathring{h}_N\mathring{\varphi}_N + \end{bmatrix}
\end{aligned}
\right\}
\quad \text{IV.4}
$$

$$
\left.\begin{array}{l}
+ \gamma_N \\
+ 2\cos\varphi_N \; \mathring{h}_N(\Omega + \mathring{\lambda}_N) \\
+ (N_N + h_N)\sin\varphi_N \; \cos\varphi_N \; \mathring{\lambda}_N(2\Omega + \mathring{\lambda}_N)
\end{array}\right]
$$

In equation IV.4, h_N is measured from the geoid, but the navigator, in applying it, assumes that it is measured from the reference ellipsoid and assumes that the geoid coincides with the ellipsoid. After going through a derivation similar to that presented in Appendix III we arrive at an equation which is the equivalent of equation III.15.

$$
\left.\begin{array}{l}
\begin{bmatrix} \ddot{h}_N \\ (N_N + h_N)\cos\varphi_N \; \ddot{\lambda}_N \\ (M_N + h_N)\ddot{\varphi}_N \end{bmatrix} = \begin{bmatrix} \ddot{h} - (M + H + \zeta)\mathring{\varphi}^2 - \\ (N + H + \zeta)\cos\varphi \; \ddot{\lambda} - \\ (M + H + \zeta)\ddot{\varphi} + 3Me^2 \sin 2\varphi \; \mathring{\varphi}^2/2W^2 + \end{bmatrix.
\\[2em]
\left.\begin{array}{l}
- (N + H + \zeta)\cos^2\varphi \; \mathring{\lambda}(2\Omega + \mathring{\lambda}) \\
- 2(M + H + \zeta)\sin\varphi \; \mathring{\varphi}(\Omega + \mathring{\lambda}) + 2\cos\varphi \; \mathring{h}(\Omega + \mathring{\lambda}) \\
+ 2\mathring{h}\mathring{\varphi} + (N + H + \zeta)\sin\varphi \; \cos\varphi \; \mathring{\lambda}(2\Omega + \mathring{\lambda})
\end{array}\right] +
\\[2em]
+ R_1(-\varphi_N)R_2(\lambda_N)(\gamma + \Delta g)\cos(\varphi + \xi) \begin{bmatrix} \cos(\lambda + \eta/\cos\varphi) \\ \sin(\lambda + \eta/\cos\varphi) \\ \tan(\varphi + \xi) \end{bmatrix} +
\end{array}\right\} \quad \text{IV.5}
$$

$$+\left[\begin{array}{l}(M_N + h_N)\dot{\varphi}_N^2 + (N_N + h_N)\cos^2\varphi_N\ \dot{\lambda}_N(2\Omega + \dot{\lambda}_N) - \gamma_N \\ 2(M_N + h_N)\sin\varphi_N\ \dot{\varphi}_N(\Omega + \dot{\lambda}_N) - 2\cos\varphi_N\ \dot{h}_N(\Omega + \dot{\lambda}_N) \\ -3M_N e^2 \sin 2\varphi_N\ \dot{\varphi}_N^2/2W_N^2 - 2\dot{h}_N\dot{\varphi}_N - \\ \quad \circ\ \circ\ \circ \\ \quad \circ\ \circ\ \circ \\ -(N_N + h_N)\sin\varphi_N\ \cos\varphi_N\ \dot{\lambda}_N(2\Omega + \dot{\lambda}_N)\end{array}\right]$$

Expanding this equation we get

$$\ddot{\lambda}_N = \frac{(\gamma + \Delta g)\cos(\varphi + \zeta)}{(N_N + h_N)\cos\varphi_N}\sin(\lambda + \eta/\cos\varphi - \lambda_N) +$$

$$+ \frac{(N + H + \zeta)\cos\varphi\ \ddot{\lambda} - 2(M + H + \zeta)\sin\varphi\ \dot{\varphi}(\Omega + \dot{\lambda})}{(N_N + h_N)\cos\varphi_N} +$$

$$+ \frac{2\cos\varphi\ \dot{h}(\Omega + \dot{\lambda}) + 2(M_N + h_N)\sin\varphi_N\ \dot{\varphi}_N(\Omega + \dot{\lambda}_N)}{(N_N + h_N)\cos\varphi_N} -$$

$$- \frac{2\cos\varphi_N\ \dot{h}_N(\Omega + \dot{\lambda}_N)}{(N_N + h_N)\cos\varphi_N}$$

Collecting terms, averaging over a time increment, and setting $H = h_N = \dot{h}_N = 0$, we obtain the final expression for the average longitudinal acceleration.

$$\left.\begin{aligned}
\ddot{\lambda}_{N\,av} = {} & \left[\frac{1}{N_N \cos\varphi_N}\right]_{(j)} \Big[\frac{1}{2}(\gamma + \Delta g)\cos(\varphi + \xi)\ \sin(\lambda + \\
& + \eta/\cos\varphi - \lambda_N) + \frac{1}{2}(N + \zeta)\cos\varphi\ \ddot{\lambda} - \\
& - (M + \zeta)\sin\varphi\ \mathring{\varphi}(\Omega + \mathring{\lambda}) + \cos\varphi\ \mathring{h}(\Omega + \mathring{\lambda}) + \\
& + M_N \sin\varphi_N\ \mathring{\varphi}_N(\Omega + \mathring{\lambda}_N)\Big]_{(j)} + \\
& + \left[\frac{1}{N_N \cos\varphi_N}\right]_{(j+1)} \Big[\frac{1}{2}(\gamma + \Delta g)\cos(\varphi + \xi)\ \sin(\lambda + \\
& + \eta/\cos\varphi - \lambda_N) + \frac{1}{2}(N + \zeta)\cos\varphi\ \ddot{\lambda} - \\
& - (M + \zeta)\sin\varphi\ \mathring{\varphi}(\Omega + \mathring{\lambda}) + \cos\varphi\ \mathring{h}(\Omega + \mathring{\lambda}) + \\
& + M_N \sin\varphi_N\ \mathring{\varphi}_N(\Omega + \mathring{\lambda}_N)\Big]_{(j+1)}
\end{aligned}\right\}\ \text{IV.6}$$

$$\begin{aligned}
(M_N + h_N)\ddot{\varphi}_N = {} & (\gamma + \Delta g)\cos(\varphi + \xi)\ \cos\varphi_N \Big[\tan(\varphi + \xi) - \\
& - \tan\varphi_N\ \cos(\lambda + \eta/\cos\varphi - \lambda_N)\Big] + \\
& + \frac{1}{2}\sin 2\varphi\Big[3e^2 M\mathring{\varphi}^2/W^2 + (N + H + \zeta)\mathring{\lambda}(2\Omega + \mathring{\lambda})\Big] + \\
& + (M + H + \zeta)\ddot{\varphi} + 2\mathring{h}\mathring{\varphi} - 2\mathring{h}_N\mathring{\varphi}_N - \\
& - \frac{1}{2}\sin 2\varphi_N\Big[3e^2 M_N\mathring{\varphi}_N^2/W_N^2 + (N_N + h_N)\mathring{\lambda}_N(2\Omega + \mathring{\lambda}_N)\Big]
\end{aligned}$$

Collecting terms, averaging over a time increment, and setting $H = h_N = \mathring{h}_N = 0$, we obtain the final expression for the average latitudinal acceleration.

$$
\left.
\begin{aligned}
\ddot{\varphi}_{N\,av} = {} & \left[\frac{1}{2M_N}\right]_{(j)} \Big[(\gamma + \Delta g)\cos(\varphi + \zeta)\, \cos\varphi_N \Big[\tan(\varphi + \zeta) - \\
& - \tan\varphi_N\, \cos(\lambda + \eta/\cos\varphi - \lambda_N)\Big] + 2\dot{h}\dot{\varphi} + \\
& + \frac{\sin 2\varphi}{2}\Big[3Me^2\dot{\varphi}^2/W^2 + (N + \zeta)\,\dot{\lambda}(2\Omega + \dot{\lambda})\Big] + \\
& + (M + \zeta)\ddot{\varphi} - \frac{\sin 2\varphi_N}{2}\Big[3M_N e^2 \dot{\varphi}_N^2/W_N^2 + \\
& + N_N \dot{\lambda}_N (2\Omega + \dot{\lambda}_N)\Big]\Big]_{(j)} + \\
& + \left[\frac{1}{2M_N}\right]_{(j+1)} \Big[(\gamma + \Delta g)\cos(\varphi + \zeta)\, \cos\varphi_N \Big[\tan(\varphi + \zeta) - \\
& - \tan\varphi_N\, \cos(\lambda + \eta/\cos\varphi - \lambda_N)\Big] + 2\dot{h}\dot{\varphi} + \\
& + \frac{\sin 2\varphi}{2}\Big[3Me^2\dot{\varphi}^2/W^2 + (N + \zeta)\,\dot{\lambda}(2\Omega + \dot{\lambda})\Big] + \\
& + (M + \zeta)\ddot{\varphi} - \frac{\sin 2\varphi_N}{2}\Big[3M_N e^2 \dot{\varphi}_N^2/W_N^2 + \\
& + N_N\, \dot{\lambda}_N (2\Omega + \dot{\lambda}_N)\Big]\Big]_{(j+1)}
\end{aligned}
\right\} \text{IV.7}
$$

Once the average accelerations in latitude and longitude are computed by means of the above equations, the determination of position at any time is the same as that developed in Appendix III.

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